# REMARKS ON THE BOX PROBLEM.

Nets Hawk Katz, Elliot Krop, Mauro Maggioni

Washington University

## §0 Introduction

In [CCW], in connection with estimates on Fourier integral operators, the following question was raised. Let

$$Q = [0,1]^n \subset \mathbb{R}^n,$$

be the unit cube in $n$ dimensions. Let $f$ be a $C^n$ function on $Q$ satisfying

$$\frac{\partial^n f}{\partial x_1 \partial x_2 \dots \partial x_n} > \Lambda.$$

What upper bound can be given on the measure of the set

$$S = \{x \in Q : |f(x)| < 1\}.$$

One can attempt to bound the measure of the set using combinatorial means. By a simple application of the fundamental theorem of calculus - one sees there is constant $K_n$ depending only on the dimension so that for no sequence of pairs

$$(x_{11}, x_{12}), (x_{21}, x_{22}), \dots, (x_{n1}, x_{n2}),$$

satisfying

$$\Pi_{i=1}^n |x_{i1} - x_{i2}| \geq \frac{1}{K_n \Lambda},$$

is it the case that the points $(x_{1\alpha_1}, \dots, x_{n\alpha_n})$ are elements of $S$ for every choice of $\alpha_1, \dots, \alpha_n$ running from 1 to 2. Another way of saying this is that $S$ does not contain the corners of

1991 *Mathematics Subject Classification.* primary 35A07 secondary 53B05.

The first author was partially supported by a National Science Foundation grant

a large box with sides in the coordinate directions. We will say throughout this note that any set satisfying this restriction satisfies the (CCW)-condition with constants $(K_n, \Lambda)$.

A few applications of Cauchy Schwarz show that a set satisfying the (CCW) condition has measure at most $O(\Lambda^{\frac{1}{2^{n-1}}-\epsilon})$ for any $\epsilon > 0$. On the other hand, this bound may not be sharp - no one knows how to find a (CCW) set that large. A very easy example of a (CCW) set is as follows, however. Let

$$f = \frac{\Lambda}{n!}(x_1 + x_2 + \dots x_n - \frac{n}{2})^N.$$

The $N$th mixed partial of $f$ is $\Lambda$. Let

$$S = \{(x_1, \dots, x_n) : |f(x_1, \dots, x_n)|\} \leq 1.$$

Clearly $S$ is the set of points $O(\Lambda^{\frac{1}{n}})$ away from a plane which cuts the cube generically. Thus clearly $|S| = O(\Lambda^{\frac{1}{n}})$. This index was observed in [CCW].

Thus we have an upper bound a little worse than $O(\Lambda^{\frac{1}{2^{n-1}}})$ and a lower bound of $O(\Lambda^{\frac{1}{n}})$. However $\frac{1}{2^{n-1}}$ becomes a lot smaller than $\frac{1}{n}$ as $n$ grows bigger than 2. The first interesting case is $n = 3$ where the former is $\frac{1}{4}$ and the latter $\frac{1}{3}$.

One can make discrete analogues of this question. Let $I_{ijk}$ be an $N \times N \times N$ tensor of 1's and 0's. We say $I$ satisfies the (E) condition provided it does not contain 1's at all the corners of *any* nontrivial box. That is $I$ satisfies the (E) condition, if for any $i_1 \neq i_2, j_1 \neq j_2, k_1 \neq k_2$, we have that

$$\Pi_{l=1}^2 \Pi_{m=1}^2 \Pi_{n=1}^2 I_{i_l j_m k_n} = 0.$$

We can ask: what is the best upper bound that can be proven for the number of 1's in any tensor $I$ satisfying the (E) condition? We refer to this as the box problem. It was first stated by Erdös [E] in the context of hypergraphs. We are interested in it mainly because of its relation to the work in [CCW].

A few application of the Cauchy Schwarz inequality shows that the number of 1's is at most $O(N^{\frac{11}{4}})$. Erdös [E] conjectured it to be sharp and it is the analogue of the $\frac{1}{4}$ above. One might ask - does one have an analogue of the lower bound with $\frac{1}{3}$? Simple numerology show that this should consist of an example with $O(N^{\frac{8}{3}})$ many 1's. Unfortunately, the best published example seems to have only $O(N^{\frac{13}{5}})$ many 1's, see ([GRS], Lemma 2.4). This state of affairs is bad - since it would seem to indicate that such examples are far from providing new results on sets with the (CCW) condition. The [GRS] construction is inefficient in that it uses probabilistic methods. In its defense, one should note that in 5 and higher dimensions ([GRS], Lemma 2.5), it starts to beat the (CCW) index (and ties it in 4 dimensions) and it generally provides evidence for Erdös' conjecture.

We try to rectify the three dimensional situation.

**Theorem.** *Let $p$ be any prime. There is a $p^3-1 \times p^3-1 \times p^3-1$ tensor $I$ satisfying the (E) condition containing $p^2(p^3-1)^2$ many 1's.*

This does not provide new results about sets with the (CCW) condition but any improvement in the index would.

We prove the theorem in Section 1.

## §1 Proof of Theorem

We begin by recalling the standard construction of a counterexample for the two dimensional problem. It is simply part of the finite projective plane.

We consider the finite field $F_q$ with $q$ elements. (It must be that $q = p^n$ for some prime $p$ and some integer $n$.) We define a $q^2-1 \times q^2-1$ matrix $I_{ij}$ consisting of 1's and 0's so that for no $i_1 \neq i_2$ and $j_1 \neq j_2$ is it the case that

$$I_{i_1 j_1} I_{i_1 j_2} I_{i_2 j_1} I_{i_2 j_2} = 1. \tag{1}$$

We shall index the $q^2-1$ rows $i$ by $F_q^2 \backslash \{(0,0)\}$, which is the plane over $F_q$ excluding the origin. We shall index the columns $j$ by the set of lines in $F_q^2$ not containing the origin. There are $q^2-1$ different such lines, given by the equations

$$ax + by = 1,$$

where $(a,b) \in F_q^2 \backslash \{(0,0)\}$. (Thus we notice a duality between the set of lines we consider and the set of points we consider.) Now we define $I_{ij} = 1$ if the point indexed by $i$ is contained in the line indexed by $j$. Now since the intersection between two different lines is at most a point, we cannot have (1) whenever $j_1 \neq j_2$ and $i_1 \neq i_2$. We produce such an $I$ with $q(q^2-1)$ many 1's, which is close to the most possible.

We now attempt to generalize this idea to three dimensions in order to prove the theorem. We shall fix a prime $p$ and do geometry in $F_p^3 \backslash \{0\}$. A plane in $F_p^3$ is an equation of the form

$$ax + by + cz = 1,$$

with

$$(a,b,c) \in F_p^3 \backslash \{0\}.$$

Notice that here we have point-plane duality. Occasionally, we shall consider $F_p^3$ to be identified with $F_{p^3}$ which is the extension of $F_p$ by an irreducible cubic. We let $r$ be the root of this cubic. Then an element of $F_{p^3}$ can be written uniquely as

$$\alpha = a + rb + r^2 c,$$

with $a,b,c \in F_p$. We say $\alpha$ is real if $b = c = 0$. Otherwise, we say $\alpha$ is complex.

We are now ready to define our construction of $I_{ijk}$ a $(p^3-1)\times(p^3-1)\times(p^3-1)$ tensor of 1's and 0's. We index $i$ and $j$ by $F_{p^3}\backslash\{0\}$ and we index $k$ by $F_p^3\backslash\{(0,0,0)\}$. We will denote multiplication in $F_{p^3}$ by $*$. To each element $\alpha \in F_{p^3}\backslash\{0\}$, we assign a plane $P_\alpha$ in $F_p^3\backslash\{(0,0,0)\}$ by saying that $P_\alpha$ is given by the equation

$$ax + by + cz = 0,$$

when

$$\alpha = a + rb + r^2c.$$

Now we define $I_{ijk}$. We let $I_{ijk} = 1$ if $k \in P_{i*j}$, and we let $I_{ijk} = 0$ otherwise.

Clearly $I_{ijk}$ has $p^2(p^3-1)^2$ many 1's. To prove the theorem, it now suffices to show that $I$ satisfies the (E)-condition. We need only show that if $i_1 \neq i_2$ and $j_1 \neq j_2$ then the cardinality of

$$A_{i_1 i_2 j_1 j_2} = P_{i_1*j_1} \cap P_{i_2*j_1} \cap P_{i_1*j_2} \cap P_{i_2*j_2}, \tag{2}$$

is at most one. Now since $A_{i_1 i_2 j_1 j_2}$ is an intersection of four planes, it is clear that it is either a plane, a line, a point, or empty. We need only exclude the first two possibilities.

Since $i_1 \neq i_2$, it must be that

$$P_{i_1*j_1} \cap P_{i_2*j_1} = L_{j_1},$$

is either empty or a line since it is the intersection of two different planes. Thus we may assume it is a line. Similarly, we may assume that

$$P_{i_1*j_2} \cap P_{i_2*j_2} = L_{j_2},$$

is a line.

Now (2) is exactly $L_{j_1} \cap L_{j_2}$. Thus to show it has cardinality at most 1, we need only show that $L_{j_1}$ and $L_{j_2}$ are not the same line. Let us suppose that they are. We call the line $L$.

Let $K_1$ be the set of all $i$ so that $P_{i*j_1}$ contains the line $L$. The set of $\alpha$ so that $P_\alpha$ contains $L$ is identified to a line $K$ in $F_p^3\backslash\{(0,0,0)\}$. Thus

$$K_1 = M_{j_1}K,$$

is also a line where $M_{j_1}$ is the linear operator on $F_p^3$ induced by multiplication in $F_{p^3}$ by $j_1$. Similarly, we define $K_2$ to be the set of all $i$ so that $P_{i*j_2}$ contains the line $L$ and

$$K_2 = M_{j_2}K.$$

By our assumption $i_1, i_2 \in K_1$ and $i_1, i_2 \in K_2$. Since $K_1$ and $K_2$ are lines, this means

$$K_1 = K_2.$$

But

$$K_1 = M_{j_1} M_{j_2^{-1}} K_2 = M_{j_1 * (j_2)^{-1}} K_2 = M_{j_1 * (j_2)^{-1}} K_1.$$

Thus the linear map $M_{j_1 * (j_2)^{-1}}$ preserves the line $K_1$.

Now since $j_1 \neq j_2$ it must be that $j_1 * (j_2)^{-1} \neq 1$. We claim that any $M_\alpha$ with $\alpha \in F_{p^3} \backslash \{0, 1\}$ cannot preserve any lines not containing the origin. This is a contradiction. Thus to prove that $I$ satisfies the (E)-condition, we need only prove the claim.

We proceed to prove the claim. There are two cases, $\alpha$ is real or $\alpha$ is complex. If $\alpha$ is real, then $\alpha = d$, where $0, 1 \neq d \in F_p$. Then $M_\alpha$ takes the line

$$ax + by + cz = 1,$$

into

$$ax + by + cz = d.$$

These are clearly different lines.

Now suppose that $\alpha$ is complex and that $M_\alpha$ preserves the line $K_1$. Then $M_\alpha$ preserves the two dimensional subspace of $F_p^3$ spanned by $K_1$. Thus the characteristic polynomial of $M_\alpha$ factors nontrivially over $F_p$. But this is impossible since the characteristic polynomial of $M_\alpha$ is an irreducible cubic over $F_p$ of which $\alpha$ is a root. Thus we have proven the claim and the theorem.

## References


[CCW] A. Carbery, M. Christ, and J. Wright, *Multidimensional Van-der-Corput and sublevel set estimates*, J. Amer. Math Soc. **12** (1999), 981–1016.

[E] P. Erdös, *On extremal problems of graphs and generalized hypergraphs*, Israel J. Math **2** (1964), 183–190.

[GRS] Gunderson, Rodl, Sidorenko, *Extremal problems for sets forming Boolean algebras and complete partite hypergraphs*, J. Comb. Th. Ser. A **88** (1999), 342–368.