

CRITICAL EXPONENT OF SHORT EVEN FILTERS AND BURT-ADELSON BIORTHOGONAL WAVELETS

MAURO MAGGIONI

ABSTRACT. We determine the critical exponent of all positive filters having an even residual of degree two and present an extension to the case of degree four; we apply these results to Burt-Adelson filters, thus determining the critical exponent of all the biorthogonal wavelets they generate. After this, we consider the problem of smoothing the dual wavelets by considering longer dual filters: we first create new wavelets by imposing an extra zero in π on the new filters and study their regularity, by determining all the critical exponents. Then we release this condition on the filters and present the results of a numerical simulation intended to maximize the Sobolev regularity.

1. INTRODUCTION

The solutions of a refinement equation in the form

$$(1.1) \quad \varphi(x) = \sum_{k=K_1}^{K_2} 2\alpha_k \varphi(2x - k)$$

with $\sum_k \alpha_k = 1$ and $\sum_k (-)^k \alpha_k = 0$, can be studied on the Fourier transform side by introducing the trigonometric polynomial

$$m_0(\xi) = \sum_{k=K_1}^{K_2} \alpha_k e^{ik\xi},$$

so that if a solution φ in $\mathbb{L}^1(\mathbb{R})$ exists, it is the compactly supported function having Fourier transform given by

$$(1.2) \quad \widehat{\varphi}(\xi) = \prod_{j=1}^{+\infty} m_0(2^{-j}\xi).$$

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In this paper we consider the case in which m_0 is factorizable as

$$(1.3) \quad \begin{aligned} m_0(\xi) &= \left(\frac{1 + e^{i\xi}}{2} \right)^N \sum_{m=0}^M a_m \cos m\xi = \\ &= \left(\frac{1 + e^{i\xi}}{2} \right)^N \mathcal{L}(\xi) \end{aligned}$$

for some $N \geq 0$, $\mathcal{L} \geq 0$, $\mathcal{L}(0) = 1$ and $\mathcal{L}(\pi) \neq 0$. The factor \mathcal{L} will be called “residual filter”.

An index of regularity for the compactly supported φ with Fourier transform given by (1.2) is the critical exponent (see [7])

$$(1.4) \quad b := \inf_j b_j := \inf_j \left\{ \log_2 \sup_{\xi \in \mathbb{R}} \left(\prod_{k=0}^{j-1} |\mathcal{L}(2^k \xi)| \right)^{\frac{1}{j}} \right\}.$$

which allows to estimate the growth of the products that define $\widehat{\varphi}$. The subadditivity of $\{b_j\}_{j \in \mathbb{N}}$ implies that actually

$$b = \lim_{j \rightarrow \infty} b_j$$

and it is known ([3]) that the sharp estimate

$$\forall \epsilon > 0 \quad |\widehat{\varphi}(\xi)| \leq C_\epsilon (1 + |\xi|)^{-N+b+\epsilon}$$

holds. Hence $-N + b$ is the Sobolev exponent s_∞ , as defined below in (1.9). It gives informations on the regularity of φ ; for example one has

$$\varphi \in \mathcal{C}^\alpha$$

for $0 \leq \alpha < N - b - 1$ and

$$\varphi \notin \mathcal{C}^\alpha$$

if $\alpha > N - b$, since φ is compactly supported.

Estimates for the critical exponent arise naturally from the consideration of invariant cycles of the map

$$(1.5) \quad \begin{aligned} \tau : S^1 &\rightarrow S^1 \\ \xi &\mapsto 2\xi \pmod{2\pi} \end{aligned} \quad ,$$

namely, if $\gamma = \{\xi_0, \dots, \xi_{n-1}\}$ is such a cycle, we have

$$(1.6) \quad b \geq \log_2 \left(\prod_{k=0}^{n-1} |\mathcal{L}(\xi_k)| \right)^{\frac{1}{n}} .$$

By considering $\gamma = \{-\frac{2\pi}{3}, \frac{2\pi}{3}\}$, we get

$$(1.7) \quad b \geq \log_2 \left| \mathcal{L}\left(\frac{2\pi}{3}\right) \mathcal{L}\left(-\frac{2\pi}{3}\right) \right|^{\frac{1}{2}} .$$

Due to their simplicity, these estimates are widely used and actually give good results. In fact, rather unexpectedly, for many important filters there exists a cycle on which (1.6) is an equality. It has been proven, by Cohen and Conze in [4] and independently by Volkmer in [13] that equality holds in (1.7) for all the minimal length filters of Daubechies. This result was used to establish the asymptotic growth of the regularity of Daubechies' compactly supported wavelets ${}_N\psi$ ([7], [13]). In [4] the (yet open) problem of establishing when there exists a cycle for which (1.6) is an equality was posed.

In this paper we proceed in the spirit of these observations. Our aim is to determine the critical exponent of "short" nonnegative residual filters

$$(1.8) \quad \mathcal{L}(\xi) = \sum_{k=0}^4 a_k \cos k\xi .$$

We notice that the family of filters $\cos^{2N}(\frac{\xi}{2}) \mathcal{L}(\xi)$ with \mathcal{L} as in (1.8) contains as a particular case the classical primal and dual Burt-Adelson filters.

We are able to prove that for "almost all" filters having residual as in (1.8) the critical exponent is attained exactly either on the trivial cycle $\{0\}$ or the shortest nontrivial cycle $\{-\frac{2\pi}{3}, \frac{2\pi}{3}\}$, thus obtaining an explicit expression for b . Moreover, all Burt-Adelson filters fall under the scope of our Theorem.

Only “few” classical Burt-Adelson wavelets have both good regularity properties and do generate biorthogonal bases; in particular some quite regular primal wavelets have non-smooth duals, or, on the contrary, smooth duals are often associated to poorly regular primal wavelets. On the other hand, in the case of Burt-Adelson wavelets the residual filter \mathcal{L} has only length 2, while our result holds for \mathcal{L} with length up to 4. The idea of increasing the regularity of the dual wavelets arises. In fact we create a new, smoother family of extended dual wavelets by adding a “tap” to the dual filter and imposing an extra zero in π . We study the improved regularity of these new wavelets by means of the critical exponent, and find conditions for obtaining biorthogonal bases.

We also compute numerically the Sobolev exponent, defined by

$$(1.9) \quad s_p(\varphi) = \sup\{s : \varphi \in \mathcal{H}_p^s\}$$

where

$$\mathcal{H}_p^s = \{f \in \mathbb{L}^p(\mathbb{R}) : \|f\|_{s,p} := \|\widehat{f}(\xi)(1 + |\xi|^2)^{\frac{s}{2}}\|_{\mathbb{L}^p(\mathbb{R})} < \infty\}$$

This will give better estimates for Hölder regularity than those given by the critical exponent. Since we will be mainly interested in the case $p = 2$, we will adopt the abuse of notation $\mathcal{H}^s = \mathcal{H}_p^s$.

Finally we consider the problem of maximizing (among the class of our extended dual filters) the Sobolev regularity. Numerical computations strongly suggest that for certain values of the parameter the choice of imposing the extra zero in π (which is an algebraic constraint) is actually the best one, while for other values the numerical results give (slightly) smoother wavelets.

For algebraic manipulations the Author used Maple, while MatLab was of much help in numerical simulations.

2. CRITICAL EXPONENT OF SHORT EVEN RESIDUAL FILTERS

The aim of this section is to find the critical exponent of even filters as in (1.3) with $M = 4$. The main result is the following Theorem. While it allows the determination of the critical exponent of every nonnegative even residual filter of degree $M = 2$, there exist nonnegative residual filters of degree 4 which do not satisfy any of its hypotheses.

Theorem 2.1. The critical exponent of a filter in the form

$$(2.1) \quad \begin{aligned} m_0(\xi) &= \left(\frac{1 + e^{i\xi}}{2} \right)^N \sum_{m=0}^4 a_m \cos m\xi = \\ &= \left(\frac{1 + e^{i\xi}}{2} \right)^N \mathcal{L}(\xi) \end{aligned}$$

with $\mathcal{L}(\xi) \geq 0$, $\mathcal{L}(0) = 1$ and $\mathcal{L}(\pi) \neq 0$ is given by

$$b = \begin{cases} 0 & \text{if } a_3 \geq 0 \text{ and } a_1 + a_2 + a_4 \geq 0 \\ \log_2 \mathcal{L}\left(\frac{2\pi}{3}\right) & \text{if } \begin{cases} a_3 \geq 0 & \text{and } a_1 + a_2 + a_4 \leq 0, \text{ or} \\ a_3 \leq 0 & \text{and } a_1 + a_2 + a_4 - 3a_3 \leq 0 \end{cases} \end{cases}$$

Proof. In order to ease the notation, let

$$\sigma_j(\xi) = \frac{1}{j} \sum_{k=0}^{j-1} \cos(2^k \xi).$$

The geometric-arithmetic means inequality gives the estimate

$$\begin{aligned} \left(\prod_{k=0}^{j-1} \mathcal{L}(2^k \xi) \right)^{\frac{1}{j}} &\leq \sum_{k=0}^4 a_k \sigma_j(2^k \xi) \\ &= a_0 + (a_1 + a_2 + a_4) \sigma_j(\xi) + a_3 \sigma_j(3\xi) + \\ &\quad + \frac{1}{j} (a_4 \cos(2^{j+1}\xi) + (a_2 + a_4)(\cos(2^j \xi) - \cos \xi) - a_4 \cos(2\xi)) \\ &= a_0 + (a_1 + a_2 + a_4) \sigma_j(\xi) + a_3 \sigma_j(3\xi) + \mathcal{O}\left(\frac{1}{j}\right) \end{aligned}$$

from which it follows that

$$(2.2) \quad \sup_{\xi \in \mathbb{R}} \left(\prod_{k=0}^{j-1} \mathcal{L}(2^k \xi) \right)^{\frac{1}{j}} \leq a_0 + \sup_{\xi \in \mathbb{R}} [(a_1 + a_2 + a_4) \sigma_j(\xi) + a_3 \sigma_j(3\xi)] + \mathcal{O}\left(\frac{1}{j}\right)$$

since the $\mathcal{O}\left(\frac{1}{j}\right)$ was uniform in ξ . It is clear that

$$\sup_{\xi \in \mathbb{R}} \sigma_j(\xi) = 1$$

but we need also to find $\inf_{\xi \in \mathbb{R}} \sigma_j(\xi)$, at least for large j . We claim that

$$(2.3) \quad \liminf_{j \rightarrow \infty} \inf_{\xi \in \mathbb{R}} \sigma_j(\xi) = -\frac{1}{2}.$$

To see this, we solve for $\cos \xi$ the identity

$$(2.4) \quad |1 + e^{i\xi} + e^{i2\xi}|^2 = 3 + 4 \cos \xi + 2 \cos 2\xi$$

and after substituting in the expression for σ_j

$$\begin{aligned} \sigma_j(\xi) &= \frac{1}{j} \sum_{k=0}^{j-1} \cos(2^k \xi) \\ &= \frac{1}{4j} \sum_{k=0}^{j-1} (|1 + e^{i2^k \xi} + e^{i2^{k+1} \xi}|^2 - 3 - 2 \cos(2^{k+1} \xi)) = \\ &= -\frac{3}{4} - \frac{1}{2j} \sum_{k=1}^j \cos(2^k \xi) + \frac{1}{4j} \sum_{k=0}^{j-1} |1 + e^{i2^k \xi} + e^{i2^{k+1} \xi}|^2 = \\ &= -\frac{3}{4} - \frac{1}{2} \sigma_j(\xi) + \frac{1}{2j} (\cos \xi - \cos 2^j \xi) + \\ &\quad + \frac{1}{4j} \sum_{k=0}^{j-1} |1 + e^{i2^k \xi} + e^{i2^{k+1} \xi}|^2 \end{aligned}$$

we get

$$\begin{aligned} \sigma_j(\xi) &= -\frac{1}{2} + \frac{1}{3j} (\cos \xi - \cos 2^j \xi) + \frac{1}{6j} \sum_{k=0}^{j-1} |1 + e^{i2^k \xi} + e^{i2^{k+1} \xi}|^2 \\ &\geq -\frac{1}{2} + \mathcal{O}\left(\frac{1}{j}\right). \end{aligned}$$

Since $\mathcal{O}(\frac{1}{j})$ is uniform in ξ , this implies the following chain of inequalities

$$-\frac{1}{2} = \sigma_j\left(\frac{2\pi}{3}\right) \geq \inf_{\xi \in \mathbb{R}} \sigma_j(\xi) \geq -\frac{1}{2} - \left| \mathcal{O}\left(\frac{1}{j}\right) \right| ,$$

and (2.3) is proved.

For $\xi \in \{-\frac{2\pi}{3}, \frac{2\pi}{3}\} \cup \{0\}$ we have $\cos 3\xi = 1$: this suggest that for $a_3 \geq 0$ we can use in (2.2) the brutal estimate

$$\begin{aligned} a_0 + \sup_{\xi \in \mathbb{R}} [(a_1 + a_2 + a_4) \sigma_j(\xi) + a_3 \sigma_j(3\xi)] + \mathcal{O}\left(\frac{1}{j}\right) &\leq \\ &\leq a_0 + a_3 + \sup_{\xi \in \mathbb{R}} (a_1 + a_2 + a_4) \sigma_j(\xi) + \mathcal{O}\left(\frac{1}{j}\right) \\ &\leq \begin{cases} a_0 + a_3 + (a_1 + a_2 + a_4) \inf_{\xi \in \mathbb{R}} \sigma_j(\xi) + \mathcal{O}\left(\frac{1}{j}\right) & \text{if } a_1 + a_2 + a_4 \leq 0 \\ a_0 + a_3 + (a_1 + a_2 + a_4) \sup_{\xi \in \mathbb{R}} \sigma_j(\xi) + \mathcal{O}\left(\frac{1}{j}\right) & \text{if } a_1 + a_2 + a_4 \geq 0 \end{cases} \\ &= \begin{cases} \mathcal{L}\left(\frac{2\pi}{3}\right) + \mathcal{O}\left(\frac{1}{j}\right) & \text{if } a_1 + a_2 + a_4 \leq 0 \\ \mathcal{L}(0) + \mathcal{O}\left(\frac{1}{j}\right) & \text{if } a_1 + a_2 + a_4 \geq 0 \end{cases} . \end{aligned}$$

If we let j go to infinity, and observe that an application of (1.6) to $\gamma = \{0\}$ and $\gamma = \{-\frac{2\pi}{3}, \frac{2\pi}{3}\}$ yields

$$(2.5) \quad b \geq \max \left\{ \log_2 \mathcal{L}\left(\frac{2\pi}{3}\right), 0 \right\} ,$$

we obtain the thesis in the case $a_3 \geq 0$.

When $a_3 \leq 0$ competition between $\sigma_j(\xi)$ and $\sigma_j(3\xi)$ arises when one tries to estimate the sup in (2.2). The idea is to use trigonometric identities like (2.4), but for $\cos 3\xi$:

$$\begin{aligned} |1 - e^{i3\xi}|^2 &= 2 - 2 \cos^3 \xi \\ \cos 3\xi &= 4 \cos^3 \xi - 3 \cos \xi \end{aligned}$$

The former leads, by proceeding as above, to the conclusion that

$$\liminf_{j \rightarrow \infty} \frac{1}{j} \sum_{k=0}^{j-1} \cos^3(2^k \xi) = -\frac{1}{8}$$

while the latter is used in the following chain of inequalities:

$$\begin{aligned}
\frac{1}{j} \sum_{k=0}^{j-1} \mathcal{L}(2^k \xi) &= a_0 + a_1 \sigma_j(\xi) + a_2 \sigma_j(2\xi) + a_3 \frac{1}{j} \sum_{k=0}^{j-1} \cos(2^k 3\xi) \\
&\quad + a_4 \sigma_j(4\xi) + \mathcal{O}\left(\frac{1}{j}\right) = \\
&= a_0 + (a_1 + a_2 + a_4 - 3a_3) \sigma_j(\xi) \\
&\quad + 4a_3 \frac{1}{j} \sum_{k=0}^{j-1} \cos^3(2^k \xi) + \mathcal{O}\left(\frac{1}{j}\right) \leq \\
&\leq a_0 + (a_1 + a_2 + a_4 - 3a_3) \sigma_j(\xi) - \frac{a_3}{2} + \mathcal{O}\left(\frac{1}{j}\right)
\end{aligned}$$

since $a_3 \leq 0$. Passing to the supremum and letting j to infinity, the right hand side tends to $a_0 - \frac{a_1 + a_2 + a_4}{2} + a_3 = \mathcal{L}\left(\frac{2\pi}{3}\right)$ if $a_1 + a_2 + a_4 - 3a_3 \leq 0$, and, using again (2.5), the Theorem is completely proved. \square

3. BURT-ADELSON'S FILTERS

Burt-Adelson's filters belong to a one-parameter family of finite, real, even filters, originally studied in view of their application to digital image processing ([2]). They cannot be fitted into the context of orthonormal wavelets, since it is well known that the only orthonormal system of wavelets having compact support and symmetry is the Haar system ([7]). However, they do fit into the biorthogonal setting, as was first noticed by M.Barlaud ([1]).

The primal filters are chosen in the form

$$m_{0,a}(\xi) = \cos^2\left(\frac{\xi}{2}\right) f_a(\xi)$$

where f_a is a polynomial of degree one in $\cos \xi$; this factorization is required in order to have a certain regularity. One looks for dual filters which are finite, real, even and again factorizable as

$$\tilde{m}_{0,a}(\xi) = \cos^2\left(\frac{\xi}{2}\right) \tilde{f}_a(\xi)$$

with \tilde{f}_a of degree two in $\cos \xi$. This choice for the degree of \tilde{f}_a is minimal in the sense that there are no solutions among shorter filters, while in this case we have a unique solution. By imposing the biorthogonality conditions

$$(3.1) \quad \overline{m_{0,a}(\xi)} \tilde{m}_{0,a}(\xi) + \overline{m_{0,a}(\xi + \pi)} \tilde{m}_{0,a}(\xi + \pi) = 1,$$

and the “high-pass” constraint

$$\tilde{m}_{0,a}(\pi) = 0$$

one finds ([1],[3]) that the remainders f_a and \tilde{f}_a can be parametrized as follows

$$(3.2) \quad \begin{aligned} f_a(\xi) &= 4a - 1 + (2 - 4a) \cos(\xi) \\ \tilde{f}_a(\xi) &= \frac{(8a^2 - 10a + 5) - (3 - 4a)^2 \cos(\xi) + (3 - 4a)(1 - 2a) \cos(2\xi)}{4a - 1} \end{aligned}$$

where, a priori, $a \in \mathbb{R} \setminus \{\frac{1}{4}\}$. The exclusion of the value $a = \frac{1}{4}$ is due to the fact that $m_{0,\frac{1}{4}}(\frac{\pi}{2}) = 0 = m_{0,\frac{1}{4}}(-\frac{\pi}{2})$ and thus the biorthogonality conditions (3.1) cannot be satisfied when $\xi = \frac{\pi}{2}$. We want to ensure that the scaling functions are in $\mathbb{L}^2(\mathbb{R})$ and generate biorthogonal unconditional systems. A well known sufficient condition, which involves only the critical exponents, is ([5],[6])

$$(3.3) \quad \begin{aligned} N - b_a &> \frac{1}{2} \\ N - \tilde{b}_a &> \frac{1}{2} \end{aligned}$$

where b_a and \tilde{b}_a are the critical exponents of $m_{0,a}$ and $\tilde{m}_{0,a}$ respectively, and N is the order of the zero in π . In our case $N = 2$ except for exceptional values of a (for example $a = \frac{3}{8}$ gives $N = 4$). Cohen studied these filters (in [3]) for particular values of the parameter, and conjectured that the critical exponent could be given in many cases by “evaluation on the cycle $\{-\frac{2\pi}{3}, \frac{2\pi}{3}\}$ ”, i.e. equality held in (1.7). Thanks

to the results of the preceding section, we can solve quite easily the problem of finding the critical exponent of all these filters.

Theorem 3.1. Let $\mathcal{I}_b^{class} = (\alpha_1, \alpha_2)$, with

$$\alpha_1 = \frac{17}{24} + \frac{\sqrt{2}}{3} - \frac{\sqrt{33 + 176\sqrt{2}}}{24} \cong 0.48$$

$$\alpha_2 = \frac{\sqrt{2} + 1}{3} \cong 0.8 .$$

Then for $a \in \mathcal{I}_b^{class}$ the critical exponent for the Burt Adelson's scaling functions φ_a and $\tilde{\varphi}_a$ and the wavelets ψ_a and $\tilde{\psi}_a$ is given by

$$b_a = \begin{cases} 0 & a \in (\alpha_1, \frac{1}{2}] \\ \log_2 f_a(\frac{2\pi}{3}) & a \in [\frac{1}{2}, \alpha_2) \end{cases}$$

$$\tilde{b}_a = \begin{cases} \log_2 \tilde{f}_a(\frac{2\pi}{3}) & a \in (\alpha_1, \frac{3}{4}] \\ 0 & a \in [\frac{3}{4}, \alpha_2) \end{cases}$$

Moreover, for these values of a $\varphi_a, \tilde{\varphi}_a, \psi_a, \tilde{\psi}_a$ are in $\mathbb{L}^2(\mathbb{R})$ and generate unconditional biorthogonal bases.

Proof. We prove that for $a \in \mathcal{I}_b^{class}$ the filters $m_{0,a}, \tilde{m}_{0,a}$ are positive, and then apply Theorem 2.1 to find the critical exponents. The last part of the thesis then follows by showing that for these values of a the conditions (3.3) are satisfied.

A simple study of the primal filters reveals that they are positive for $a > \frac{3}{8}$ (it is enough to observe their monotonicity on $[0, \pi]$), and their critical exponent is then found immediately by applying Theorem 2.1. To satisfy the first inequality of (3.3) we have thus to impose

$$\log_2 f_a\left(\frac{2\pi}{3}\right) = \log_2(6a - 2) < \frac{3}{2}$$

only for $a \geq \frac{1}{2}$; we conclude that for $a \in (\frac{3}{8}, \frac{1+\sqrt{2}}{3})$ the first condition in (3.3) is satisfied.

We now turn to the dual filters: we study first their positivity. An evaluation at π gives

$$\tilde{f}_a(\pi) = \frac{32a^2 - 44a + 17}{4a - 1}$$

which is positive for $a > \frac{1}{4}$, so we restrict the parameter to this interval. Simple calculations show that the minimum of the residual filter is attained at a point different from 0 and π when $a \geq \frac{7}{12}$. The value of the filter at such a point is

$$-\frac{1}{8} \frac{16a^2 - 32a + 11}{-1 + 2a}$$

which is positive if $a \in [\frac{7}{12}, 1 + \frac{\sqrt{5}}{4})$. When $\frac{1}{4} < a < \frac{7}{12}$ the minimum is attained at 0 or at π and is thus positive, since we have already restricted a so that $\tilde{f}_a(\pi) \geq 0$. In conclusion the dual filters are positive for $a \in (\frac{1}{4}, 1 + \frac{\sqrt{5}}{4})$. An application of Theorem 2.1 yields immediately the critical exponent. To satisfy the second inequality in (3.3) we have to solve

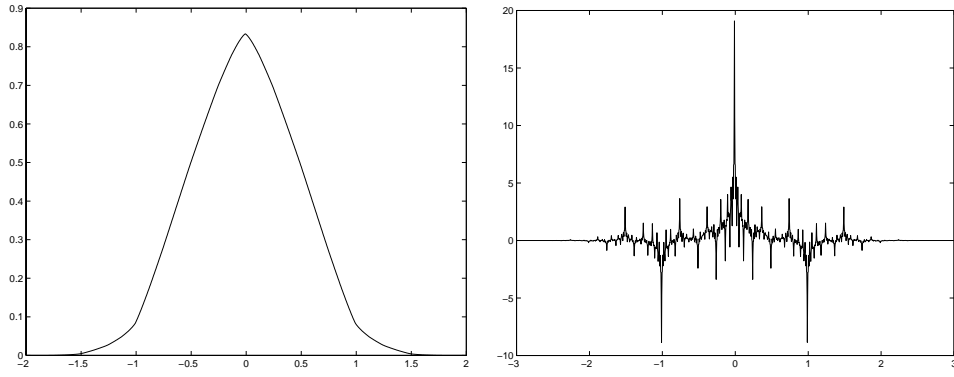
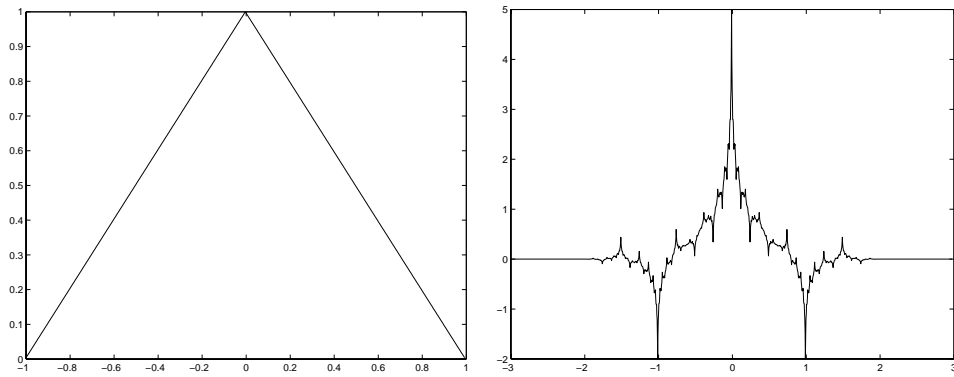
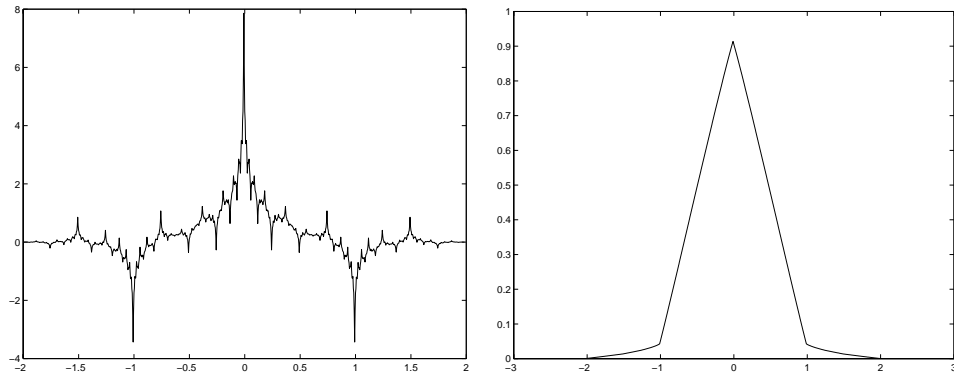
$$\tilde{b}_a = \log_2 \tilde{f}_a\left(\frac{2\pi}{3}\right) = \log_2 \frac{12a^2 - 17a + 8}{4a - 1} < \frac{3}{2}$$

only when $a \in (\frac{1}{4}, \frac{3}{4})$. This is straightforward and leads to

$$a \in \left(\frac{17}{24} + \frac{\sqrt{2}}{3} - \frac{\sqrt{33 + 176\sqrt{2}}}{24}, \frac{17}{24} + \frac{\sqrt{2}}{3} + \frac{\sqrt{33 + 176\sqrt{2}}}{24} \right)$$

Putting together the conditions on a we have that for $a \in \mathcal{I}_b^{class}$ both the conditions in (3.3) are satisfied, and the last part of thesis follows. \square

The conditions (3.3) were substantially introduced to ensure that the scaling functions are in $\mathbb{L}^2(\mathbb{R})$, which is a fundamental ingredient in proving the biorthogonality property. They are only sufficient conditions; a well-known condition that characterizes the stability of

FIGURE 1. $\varphi_{\frac{45}{10}}$ and $\tilde{\varphi}_{\frac{45}{10}}$ FIGURE 2. $\varphi_{\frac{5}{10}}$ and $\tilde{\varphi}_{\frac{5}{10}}$ FIGURE 3. $\varphi_{\frac{8}{10}}$ and $\tilde{\varphi}_{\frac{8}{10}}$

biorthogonal wavelet bases is due to Cohen and Daubechies ([5]), and

is essentially based on the transition operator

$$T_{r_a} : \mathcal{C}([0, \pi]) \longrightarrow \mathcal{C}([0, \pi])$$

defined by

$$T_{r_a}(u)(\xi) = r_a\left(\frac{\xi}{2}\right)u\left(\frac{\xi}{2}\right) + r_a\left(\pi - \frac{\xi}{2}\right)u\left(\pi - \frac{\xi}{2}\right)$$

where r_a is the residual

$$r_a(\xi) = \frac{|m_{0,a}(\xi)|^2}{(2 + 2 \cos \xi)^2} .$$

It is also well known (see ([5]), [10], [12], [11]) that the spectral radius μ_a of this operator is related to the Sobolev exponent s_2 via

$$s_2(\varphi_a) = -\log_4 \mu_a ,$$

and that this spectral radius is the same as that of the restriction

$T_{r_a}|_{\mathcal{P}_L}$, where

$$\mathcal{P}_L = \left\{ \sum_{k=0}^L c_k \cos(k\xi) : \{c_k\}_k \in \mathbb{R} \right\} ,$$

L being the degree of r_a .

As an application of these results, we computed numerically the Sobolev exponent s_2 of Burt-Adelson wavelets, and compared graphically (Fig.4) the results with those given by the critical exponent, keeping in mind on the inequalities

$$(3.4) \quad N - b - \frac{1}{2} \leq s_2(\varphi) \leq N - b .$$

The results in [5] imply that in this way we can obtain sharp (numerical) conditions for having biorthogonal wavelet bases. In this way we can estimate that the admissible values of a for which the Burt-Adelson filters generate biorthogonal bases are

$$(3.5) \quad a \in \mathcal{I}_{sob}^{class} \approx (0.447358, 0.916503) .$$

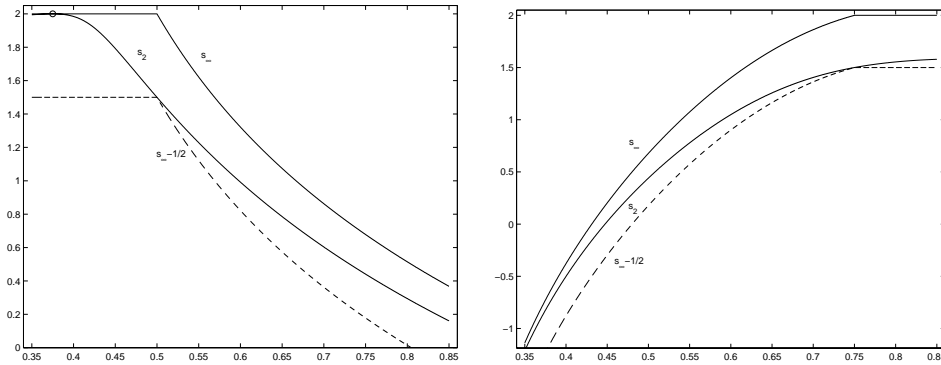


FIGURE 4. Regularity of primal (on the left) and dual (on the right) wavelets as given by the s_∞ and s_2 exponents

Estimates for Hölder regularity can be deduced by the inclusions

$$(3.6) \quad \mathcal{H}^s \subset \mathcal{C}^n \quad \text{for} \quad n + \frac{1}{2} < s .$$

Remark 3.1. Among Burt-Adelson's primal scaling functions there are some box splines of low degree. In particular for $a = \frac{1}{2}$ one has the piecewise linear spline and for $a = \frac{3}{8}$ the piecewise cubic spline.

The B-spline biorthogonal wavelets were constructed by Cohen, Daubechies and Faveau in [6], where they find the following parametrization for the primal even filters

$$(3.7) \quad {}_N m_0(\xi) = \cos^N(\xi/2)$$

and for the corresponding duals

$$(3.8) \quad {}_{N, \tilde{N}} \tilde{m}_0(\xi) = \cos^{\tilde{N}}(\xi/2) \left[\sum_{n=0}^{k-1} \binom{k-1+n}{n} \sin^{2n}(\xi/2) + \sin^{2k}(\xi/2) R(\cos \xi) \right]$$

where $N + \tilde{N}$ is even and R is an odd polynomial. The Burt-Adelson biorthogonal filters with $a = \frac{1}{2}$ and $a = \frac{3}{8}$ correspond respectively to the pairs $({}_2 m_0; {}_2 \tilde{m}_0)$ and $({}_4 m_0; {}_4 \tilde{m}_0)$, with $R = 0$. In particular we see that ${}_4 \tilde{m}_0$ does not generate an $L^2(\mathbb{R})$ dual.

For $a = \frac{3}{8}$ (cubic spline wavelet) we have $s_2(\varphi_{\frac{3}{8}}) = \frac{7}{2}$ (as one can easily calculate after definition (1.9)), and there is a jump in Sobolev regularity. More precisely, for every $\epsilon > 0$ we have $2 - \epsilon < s_2(\varphi_a) < 2$ for $a \in \mathcal{U}_\epsilon(\frac{3}{8}) \setminus \{\frac{3}{8}\}$. This discontinuity is due to the “exceptional” circumstance that for $a = \frac{3}{8}$ two more zeros appear in π , and means that this spline scaling function is “approached” with scaling functions which are significantly less regular, at least in Sobolev sense. A similar phenomenon was acknowledged by Villemoes while studying another family of filters (see [12]).

4. SMOOTHING BURT-ADELSON’S DUAL WAVELETS

The purpose of this section is to construct new, smoother, dual wavelets for the Burt-Adelson filters. We use the estimates of regularity derived in the preceding sections to find the critical exponent of these new dual wavelets; we also compute their Sobolev regularity by means of the spectral radii of the corresponding transition operators and discuss the improvement in regularity achieved by passing from the classical to the new wavelets.

Let us recall the expression of the primal filter:

$$(4.1) \quad \begin{aligned} m_{0,a}(\xi) &= \cos^2\left(\frac{\xi}{2}\right) f_a(\xi) \\ f_a(\xi) &= 4a - 1 + (2 - 4a) \cos \xi \end{aligned}$$

which can be written in the form $m_{0,a}(\xi) = \sum_{l=-2}^2 h_l e^{-il\xi}$ with coefficients (“taps”)

$$(h_0, h_{\pm 1}, h_{\pm 2}) = \left(a, \frac{1}{4}, \frac{1}{4} - \frac{a}{2}\right).$$

We will look for a dual even filter in the form

$$\tilde{m}_{0,a}(\xi) = \cos^2\left(\frac{\xi}{2}\right) \tilde{f}_a(\xi)$$

where \tilde{f}_a is a polynomial in $\cos \xi$ of degree four. This means we have 11 coefficients $\tilde{h}_0, \dots, \tilde{h}_{\pm 5}$. We impose the biorthogonality conditions

$$2 \sum_{l=-5}^5 h_l \tilde{h}_{l+2n} = \delta_{0,n} \quad \forall n \in \mathbb{Z}$$

and the constraint

$$(4.2) \quad \tilde{m}_{0,a}(\pi) = 0$$

so that $\tilde{m}_{0,a}$ is divisible by $\cos^2 \frac{\xi}{2}$. Taking into account the relation $\tilde{h}_k = \tilde{h}_{-k}$, we're left with the linear system

$$\begin{pmatrix} a & \frac{1}{2} & \frac{1}{2} - a & 0 & 0 & 0 \\ \frac{1}{4} - \frac{1}{2}a & \frac{1}{4} & a & \frac{1}{4} & \frac{1}{4} - \frac{1}{2}a & 0 \\ 0 & 0 & \frac{1}{4} - \frac{1}{2}a & \frac{1}{4} & a & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} - \frac{1}{2}a & \frac{1}{4} \\ 1 & -2 & 2 & -2 & 2 & -2 \end{pmatrix} \begin{pmatrix} \tilde{h}_0 \\ \tilde{h}_1 \\ \tilde{h}_2 \\ \tilde{h}_3 \\ \tilde{h}_4 \\ \tilde{h}_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which has a one-parameter family of solutions (in the classical case we had an unique solution) which can be parametrized as follows:

$$\begin{pmatrix} \tilde{h}_0 \\ \tilde{h}_{\pm 1} \\ \tilde{h}_{\pm 2} \\ \tilde{h}_{\pm 3} \\ \tilde{h}_{\pm 4} \\ \tilde{h}_{\pm 5} \end{pmatrix} = \begin{pmatrix} -(-4a - 1 + 32a\nu - 8\nu)/[4(4a - 1)] \\ (-8a^2 + 18a - 5 + 128a^2\nu - 32a\nu)/[8(4a - 1)] \\ (4a - 3)/[8(4a - 1)] \\ -(-8a^2 + 10a - 3 + 192a^2\nu - 80a\nu + 8\nu)/[8(4a - 1)] \\ \nu \\ (2a - 1)\nu \end{pmatrix}$$

The classic solution is easily recognized when $\nu = 0$ (which gives no terms in $\cos 3\xi$ and $\cos 4\xi$). The expression of the filter in the frequency domain is apparent from the taps; we are more interested in collecting the term $\cos^2 \frac{\xi}{2}$ (which exists by the constraint in (4.2)):

$$\tilde{m}_{0,a,\nu}(\xi) = \cos^2 \left(\frac{\xi}{2} \right) \tilde{f}_{a,\nu}(\xi)$$

with

$$\begin{aligned}
(4.3) \quad \tilde{f}_{a,\nu}(\xi) = & \{ 2\nu(32a^2 - 24a + 4) \cos 4\xi + \\
& + 8\nu(-16a^2 + 16a - 3) \cos 3\xi + \\
& + ((3 - 4a)(1 - 2a) + 32\nu(1 - 4a)) \cos 2\xi + \\
& + (-(3 - 4a)^2 + 8\nu(16a^2 + 16a - 5)) \cos \xi + \\
& + (8a^2 - 10a + 5 + 8\nu(-8a^2 - 10a + 3)) \} \frac{1}{4a - 1}
\end{aligned}$$

4.1. Smoother wavelets obtained by adding a zero in π . We can now choose ν in order to obtain desired properties. Our goal is to obtain an improved regularity. This can be done, but this is only a way among possibly many, by adding an extra zero at π , i.e. by imposing the constraint

$$(4.4) \quad \nu = \bar{\nu}(a) = \frac{1}{128} \frac{32a^2 - 44a + 17}{4a - 1}.$$

This allows us to collect one more term $\cos^2 \frac{\xi}{2}$ in $\tilde{m}_{0,a}(\xi)$, which can thus be rewritten in the form

$$\tilde{m}_{0,a,\bar{\nu}(a)}(\xi) = \cos^4 \left(\frac{\xi}{2} \right) g_a(\xi)$$

with

$$\begin{aligned}
(4.5) \quad g_a(\xi) = & \{ (64a^3 - 120a^2 + 78a - 17) \cos 3\xi + \\
& + (-256a^3 + 512a^2 - 356a + 85) \cos 2\xi + \\
& + (448a^3 - 904a^2 + 650a - 173) \cos \xi + \\
& + (-256a^3 + 512a^2 - 356a + 101) \} \frac{1}{16a - 4}
\end{aligned}$$

The positivity of these residual filters is essential in view of applying the results of the previous section. We state it in the following Lemma.

Lemma 4.1. The residual filters g_a defined above in (4.5) are positive for

$$a \in \left[\frac{7}{20}, \frac{17}{20} \right] \supset \mathcal{I}_b^{class}.$$

Proof. We begin by observing that a simple study reveals that $g_a(\pi)$ is positive for the values of a we are considering, hence the positivity can fail only if there is a negative minimum in $(0, \pi)$.

The derivative of the residual filters g_a is

$$\frac{d}{d\xi} g_a(\xi) = C(a) \sin \xi P_a(\cos \xi)$$

where $C(a)$ is continuous function of a which doesn't vanish, while P_a is a polynomial of degree two with coefficients depending continuously on a . We show in a moment that it has at most one root in $[-1, 1]$. Suppose first that it is so. If a is such that no root of P_a is in $[-1, 1]$, then g_a is monotone on $[0, \pi]$, attains its minimum at 0 or at π , and is thus positive. If a is such that a root in $[-1, 1]$ does exist, let it be $\cos \xi_a$. By symmetry, we can study g_a only on $[0, \pi]$. A straightforward computation shows that the second order derivative of g_a is positive in 0 for $a < \frac{17}{20}$. Thus g_a is growing on the right of zero, till it reaches the flat point ξ_a : such a point cannot be a minimum, so the minimum is again attained at 0 or π , and the thesis follows.

The claim above, asserting that P_a has at most one root in $[-1, 1]$, can be verified by a direct computation of the roots of P_a : the greatest root is

$$\frac{256a^3 - 85 + 356a - 512a^2 + \sqrt{R(a)}}{4(-120a^2 - 17 + 78a + 64a^3)}$$

with

$$R(a) = 16384a^6 - 94528a^3 + 108800a^4 - 65536a^5 + \\ + 1003 - 10756a + 44768a^2$$

This root is a monotone decreasing function of a on the interval we are considering. An evaluation in $a = \frac{17}{20}$ shows that there it is greater than 1, and the claim follows. \square

The zero we added in π tends to improve the regularity, but this improvement competes with possibly larger critical exponents for the remainder g_a . We now want to determine the critical exponent of these new wavelets. The following Lemma tells us that Theorem 2.1 in the last Section is enough.

Lemma 4.2. Let g_a be as in (4.5) and let $c_0(a), \dots, c_3(a)$ be such that

$$g_a(\xi) = \sum_{k=0}^3 c_k(a) \cos k\xi.$$

Then the following inequalities are satisfied:

$$\begin{aligned} \text{for } a \in \left(\frac{7}{20}, \frac{1}{2} \right] & \quad \begin{cases} c_3(a) \leq 0 \\ c_1(a) + c_2(a) \leq 3c_3(a) \end{cases} \\ \text{for } a \in \left[\frac{1}{2}, \frac{17}{20} \right) & \quad \begin{cases} c_3(a) \geq 0 \\ c_1(a) + c_2(a) \leq 0 \end{cases} \end{aligned}$$

Proof. The explicit expression of $c_3(a)$ is

$$c_3(a) = \frac{64a^3 - 120a^2 + 78a - 17}{16a - 4}$$

The numerator has exactly one real zero $a = \frac{1}{2}$, and, since it tends to $+\infty$ as $a \rightarrow +\infty$, it is positive on $(\frac{1}{2}, +\infty)$. The signum of $c_3(a)$ is thus as claimed. For the second estimate, we observe that the numerator of

$$c_1(a) + c_2(a) = \frac{96a^3 - 196a^2 + 197a - 44}{-2(4a - 1)}$$

has only one real root, which is on the left of $7/20$ since an easy computation shows that $c_1(7/20) + c_2(7/20) \geq 0$. Hence

$$c_1(a) + c_2(a) \leq 0$$

for $a \geq 1/2$. We have now to show that

$$c_1(a) + c_2(a) \leq 3c_3(a)$$

for $a \in [7/20, 1/2)$, where $c_3(a) \leq 0$. We first compute

$$\frac{d}{da} [c_1(a) + c_2(a) - 3c_3(a)] = -2 \frac{16a^2 - 8a - 11}{(4a - 1)^2}$$

and deduce that this derivative is positive on $(1/4, 1/4 + \sqrt{3}/2) \supset [7/20, 1/2)$. Since a straightforward computation gives

$$c_1\left(\frac{1}{2}\right) + c_2\left(\frac{1}{2}\right) - 3c_3\left(\frac{1}{2}\right) < 0$$

we have the requested relation. \square

We can now prove

Theorem 4.3. The decay of the Fourier transform of the extended dual scaling functions and wavelets is

$$|\widehat{\varphi}_{a, \bar{v}(a)}(\xi)| \leq C_\epsilon (1 + |\xi|)^{-4 + \bar{b}_a + \epsilon}$$

with

$$(4.6) \quad \bar{b}_a = \log_2 |g_a(\frac{2\pi}{3})| \quad a \in (\frac{35}{10}, \frac{\sqrt{2}+1}{3})$$

For $a \in \mathcal{I}_b^{ext} := (\frac{17}{40}, \frac{\sqrt{2}+1}{3}) \supset \mathcal{I}_b^{class}$, the classical primal filters $m_{0,a}$ defined in (4.1) and the extended dual Burt-Adelson filters $\tilde{m}_{0,a, \mathcal{P}(a)}$ defined in (4.5) generate scaling functions that are in $\mathbb{L}^2(\mathbb{R})$ and give raise to biorthogonal unconditional bases.

Proof. The two Lemmas show that we can apply Theorem 2.1 in order to determine the critical exponent of all our extended dual filters.

To obtain the second part of the thesis, we impose the conditions (3.3) on the new dual filters. The first inequality in (3.3), regarding the primal scaling functions, gives immediately $\sup \mathcal{I}_b^{ext} = \sup \mathcal{I}_b^{class}$ as in the proof of Theorem 3.1.

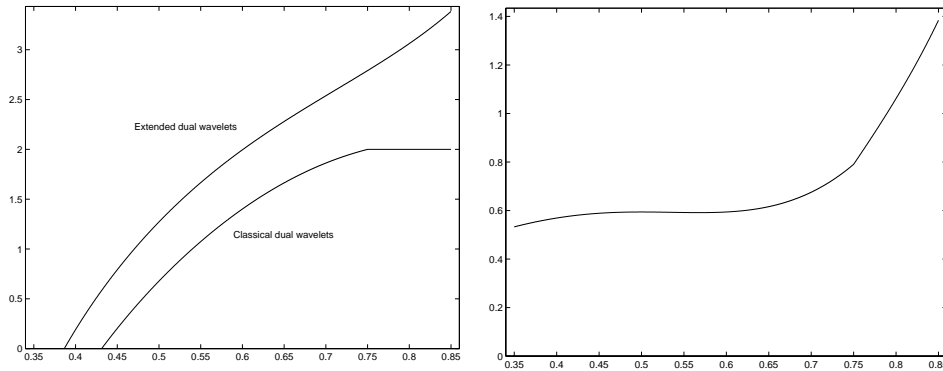


FIGURE 5. Comparison between s_∞ for classical and extended dual wavelets (on the left); gain for s_∞ achieved by passing to the new dual wavelets (on the right)

The second condition is equivalent to

$$\log_2 \frac{-288a^3 + 588a^2 - 425a + 128}{16a - 4} < 4 - \frac{1}{2}.$$

A simple study reveals that this inequality is satisfied for $a < \frac{1}{4}$ and $a > \bar{a}$. The value of \bar{a} can be determined algebraically, but it is quite cumbersome; it is clear that $\bar{a} > \frac{17}{40}$, since $\log_2 g_{\frac{17}{40}}(\frac{2\pi}{3}) < \frac{7}{2}$; actually $\bar{a} \approx \frac{17}{40}$.

Finally, \mathcal{I}_b^{ext} is obtained by intersecting the intervals of admissibility of the primal and extended dual filters. \square

In the plots in Fig.5 we confront these critical exponents with those of the classical Burt-Adelson dual wavelets. The improvement in regularity is apparent, especially for values of a greater than $\frac{3}{4}$.

We confront in Fig.6a critical exponents and Sobolev exponents (computed via the transition operators). The Sobolev regularity of the new wavelets vs. that of the classical wavelets is plotted in Fig.6b; the improvement of the Sobolev exponent is greater than that of the critical exponent.

We summon some observations on these new wavelets.

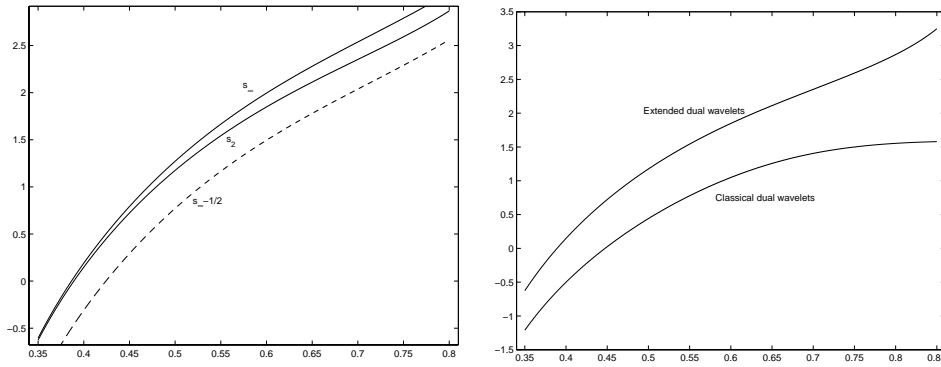


FIGURE 6. Regularity of extended dual wavelets given by s_∞ and s_2 exponent (on the left); Sobolev regularity for the new and classical dual wavelets (on the right)

- We have enlarged the class of admissible pairs of wavelets. This follows immediately by comparison between the intervals \mathcal{I}_b^{class} and \mathcal{I}_b^{ext} . By considering the Sobolev regularity we can obtain sharp (numerical) conditions: let $\mathcal{I}_{sob}^{class}$ be as in (3.5), and let \mathcal{I}_{sob}^{ext} be its analogous for the extended wavelets, i.e. let it be the maximal interval among the intervals I such that $a \in I$ implies that $\psi_{a,\overline{\nu}(a)}, \tilde{\psi}_{a,\overline{\nu}(a)}$ generate biorthogonal wavelet bases. \mathcal{I}_{sob}^{ext} can be estimated numerically, for example as

$$\mathcal{I}_{sob}^{ext} \approx (0.400, 0.916503) .$$

We recollect here our estimates of admissibility, for a convenient comparison

$$\mathcal{I}_b^{class} \approx (0.481, 0.805)$$

$$\mathcal{I}_{sob}^{class} \approx (0.477, 0.916)$$

$$\mathcal{I}_b^{ext} \approx (0.425, 0.805)$$

$$\mathcal{I}_{sob}^{ext} \approx (0.400, 0.916).$$

The new interval of admissibility \mathcal{I}_{sob}^{ext} gives completely new pairs of wavelets which generate biorthogonal bases. Interesting new pairs are those associated to values of a near the infimum

of that interval, since the primal wavelet is rather regular, and those corresponding to a near the supremum of \mathcal{I}_{sob}^{ext} , since there we have quite regular dual wavelets.

- The gain in Sobolev regularity is conspicuous, especially for $a > 1/2$. We obtain wavelets which are \mathcal{C}^2 and more for, say, $a > 3/4$, which are associated to poorly regular primal wavelets. Remember that the primal wavelets have changed too, and if the dual wavelet gains one derivative, the primal one gains one vanishing moment.

In applications regularity is sought for the reconstructing wavelet, while vanishing moments are required for the analyzing wavelet. For small values of a we have obtained new biorthogonal wavelets, with rather regular primal wavelets. For larger values of a the dual wavelets are quite smooth, while some of the corresponding irregular primal wavelets have gained one vanishing moment. It is expectable that good results could be obtained by employing these wavelets in signal analysis.

- Since our $\tilde{m}_{0,\overline{\pi}(3/8)}$ is exactly the dual filter ${}_{4,4}\tilde{m}_0$ of Cohen, Daubechies and Faveau (see [6] and (3.8), where again $R = 0$), we see, in particular, that such a filter doesn't generate a scaling function in $\mathbb{L}^2(\mathbb{R})$.

4.2. Maximal regularity for extended dual Burt-Adelson's wavelets. Nothing assures that the choice of adding an extra zero in π is the best one in order to obtain better regularity properties. Undoubtedly an advantage of such a choice is the algebraicity of the constraint, which is in general appreciated in a theoretical setting, but can

also be of help in implementations. Recent constructions have shown the possibilities and the limits of similar algebraic constraints. Heller ([11]) constructed new families of wavelets imposing extra zeroes in points that are periodic or pre-periodic for the map (1.5), such as π or $\frac{2\pi}{3}$ or $\frac{\pi}{3}$, and generalized this technique to the case of M-band wavelets. These simulations show that this technique gives good results, but is not always optimal from the point of view of regularity; families of maximally smooth wavelets (given a fixed filter length) have been constructed numerically, but it is not known whether these families could be described algebraically.

Here we consider the problem of finding the most (Sobolev) regular wavelets among those generated by our extended filters $\tilde{m}_{0,a,\nu}$ and present the results of a numerical simulation.

We release the constraint (4.2), to have back the free parameter ν , and aim at choosing $\nu = \nu^*(a)$ in such a way that for each fixed a , the scaling function $\tilde{\varphi}_{a,\nu^*(a)}$ is the smoothest among those of the family $\{\tilde{\varphi}_{a,\nu}\}_{a,\nu \in \mathbb{R}}$.

This extremal problem is treated numerically in the obvious way: we compute (adaptively) the Sobolev exponent $s_{a,\nu}$ for the general $\tilde{\varphi}_{a,\nu}$ (generated by the $\tilde{m}_{0,a,\nu}$ in (4.3)), then, for each a , we choose the $\nu^*(a)$ which maximizes the regularity.

We plot in Fig.7 the surface $s_{a,\nu}$ representing the Sobolev regularity of the scaling functions $\tilde{\varphi}_{a,\nu}$.

The conclusion is that the maximal regularity is achieved by the family $\tilde{\varphi}_{a,\nu^*(a)}$ where the parameter $\nu^*(a)$ is determined numerically

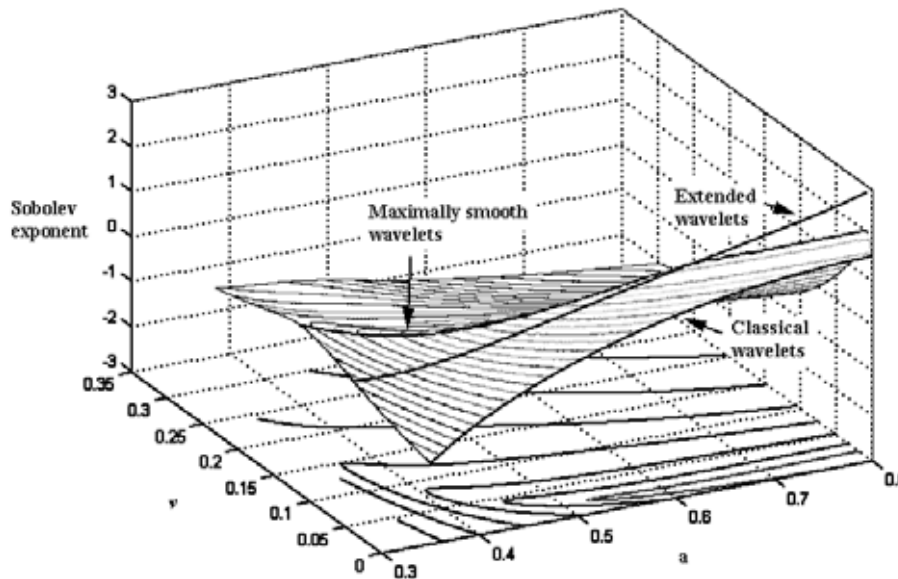


FIGURE 7. Maximization of Sobolev regularity. The plot represents the surface $s_{a,\nu}$ on which lie the lines representing the Sobolev regularity of the classical, extended and maximally smooth dual wavelets.

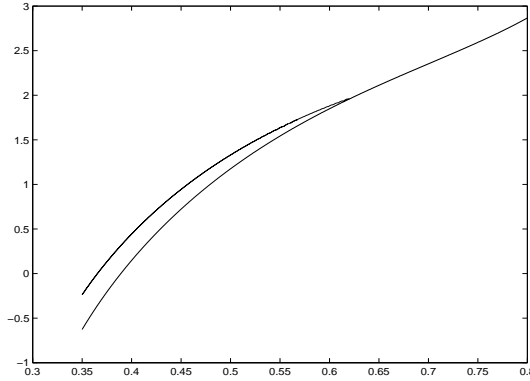


FIGURE 8. Comparison between Sobolev regularity of extended dual and maximally regular wavelets.

for $a \in (0.3, a^*)$ (with $a^* \cong 0.6$), but is exactly the $\bar{\nu}(a)$ of (4.4), the one that gives an extra zero in π , for $a \in (a^*, 0.8)$.

We confront in Fig.8 the Sobolev exponents of the extended dual vs. the maximally smooth wavelets. The improvement in regularity is appreciable for small values of a .

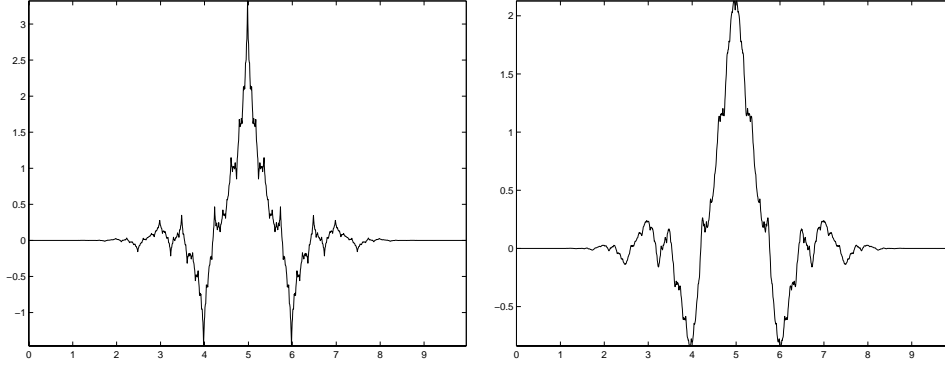


FIGURE 9. $\tilde{\varphi}_{\frac{45}{10}, \mathcal{D}(\frac{45}{10})}$ and $\tilde{\varphi}_{\frac{45}{10}, \mathcal{D}^*(\frac{45}{10})}$

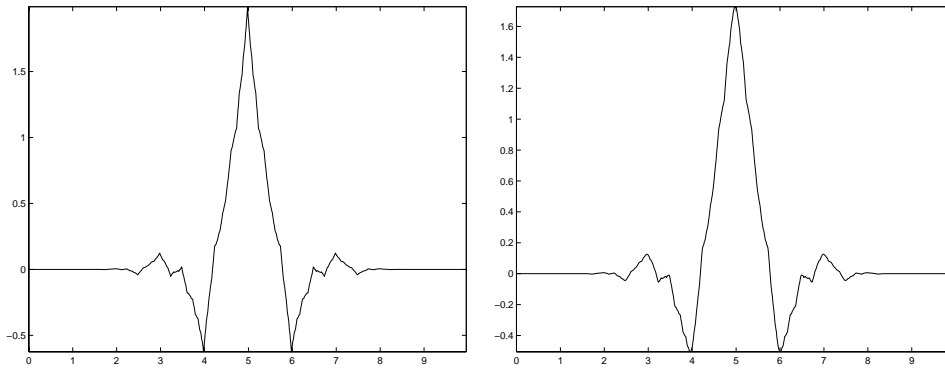


FIGURE 10. $\tilde{\varphi}_{\frac{5}{10}, \mathcal{D}(\frac{5}{10})}$ and $\tilde{\varphi}_{\frac{5}{10}, \mathcal{D}^*(\frac{5}{10})}$

Remark 4.1. It may be interesting to observe that in this way we find a new dual wavelet in $\mathbb{L}^2(\mathbb{R})$ for the piecewise cubic B-spline wavelet, with a filter shorter than the ones currently used (but also with less vanishing moments). This new maximally smooth dual corresponds to

$$\nu^*\left(\frac{3}{8}\right) \approx 0.121706987$$

and has Sobolev regularity $s_2 \approx 0.13264$.

With the notation of [6] (see the above formulas (3.7) and (3.8)), we have found a filter ${}_{4,2}\tilde{m}_0$ with $R \neq 0$, optimal from the point of view of regularity.

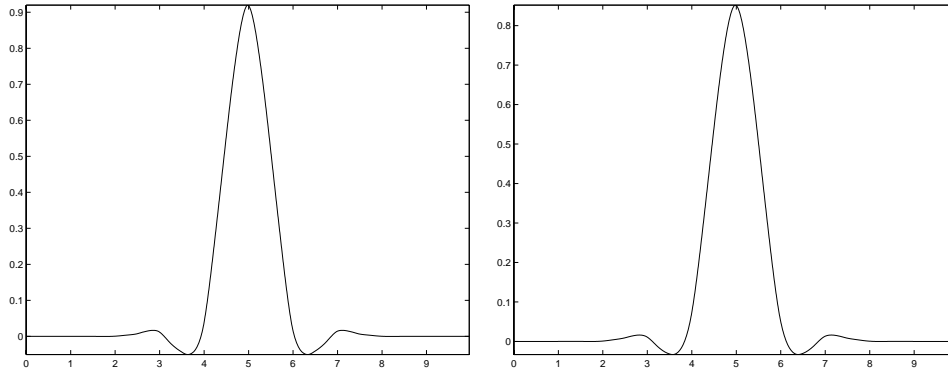


FIGURE 11. $\tilde{\varphi}_{\frac{75}{10}, \bar{\nu}(\frac{75}{10})}$ and $\tilde{\varphi}_{\frac{8}{10}, \bar{\nu}(\frac{8}{10})}$

Finally, we plot some extended and maximally smooth scaling functions, for a naked-eye comparison with the plots of classical Burt-Adelson's scaling functions (see Fig.1,2,3). Every plot is accompanied by a label indicating the scaling function displayed. Recall that $\bar{\nu}(a)$ gives the extended dual scaling function (corresponding an extra zero in π for the filter) while $\nu^*(a)$ gives the maximal smoothness.

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This paper is dedicated to my family.

REFERENCES

- [1] M.Antonini, M.Barlaud, I.Daubechies and P.Mathieu, Image coding using wavelet transform, *IEEE Trans. on Image Processing*, 1,205-220,1992
- [2] P. Burt and E. Adelson, The Laplacian pyramid as a compact image code, *IEEE Trans.Comm.*, 31, 482-540,1993
- [3] A. Cohen, Ondelettes et Traitement Numérique du Signal, Paris, Masson,1992
- [4] A. Cohen, J.P.Conze, Régularité des bases d'ondelettes et mesures ergodiques, *Rev. Math.Iberoamericana*, 8, 351-366
- [5] A. Cohen and I. Daubechies, A Stability Criterion for Biorthogonal Wavelet Bases and their Related Subband Coding Scheme, *Duke Mathematical Journal*, Vol. 68 no. 2, November 1992

- [6] A.Cohen, I.Daubechies and J.-C.Faveau, Biorthogonal Bases of Compactly Supported Wavelets, *Communications on Pure and Applied Mathematics*, Vol. XLV, 1992, pp.485-560
- [7] I. Daubechies, Ten lectures on wavelets, *CBM-NSF Regional Conference Series in Applied Math. SIAM (1992)*.
- [8] I. Daubechies and J.C. Lagarias, Two-scale Difference Equations I. Existence and global regularity of Solutions., *SIAM J.Math.Anal.*, 22 (1991),1388-1410
- [9] I. Daubechies and J.C. Lagarias, Two-scale Difference Equations II. Infinite matrix products, local regularity and fractals., *SIAM J.Math.Anal.*, 23 (1992),1031-1079
- [10] T.Eirola, Sobolev characterization of solutions of dilation equations, *SIAM J.Math.Anal.*, 23, 1015-1030,1992
- [11] P.N. Heller and R.O. Wells, Sobolev Regularity for Rank M Wavelets, preprint
- [12] L.F. Villemoes, Energy moments in time and frequency for two-scale difference equation solutions and wavelets, *SIAM J. Math. Anal.*, 23 (1992), pp.1519-1543
- [13] H. Volkmer, *Asymptotic regularity of compactly supported wavelets*, *SIAM J. Math. Anal.*, 26 (1995), pp.1075-1087.

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI MILANO-BICOCCA, EDIFICIO U7, V.LE SARCA 202, 20126 MILANO, ITALY

E-mail address: maggioni@math.wustl.edu, www.math.wustl.edu/~maggioni