M-BAND BURT-ADELSON BIORTHOGONAL WAVELETS

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Abstract. For every integer $M \geq 2$ we introduce a new family of biorthogonal MRA's with dilation factor M, generated by symmetric scaling functions with small support. This construction generalizes Burt-Adelson biorthogonal 2-band wavelets. For $M \in \{3, 4\}$ we are able to find simple explicit expressions for two different families of wavelets associated with these MRA's: one with better localization, the other with interesting symmetry/antisymmetry properties. We study the regularity of our scaling functions by determining their Sobolev exponent, for every value of the parameter and every M. We also study the critical exponent when $M = 3$.

1. Introduction

In this paper we present the construction of new families of compactly supported biorthogonal scaling functions with dilation factor M, which are symmetric and have small support. The interest in MRA structures with dilation factor greater than 2 ([4],[14],[15],[23],[30]) is motivated by the theory of M-band channel subband coding schemes ([3],[14],[15],[27]), and by the attempt to obtain greater flexibility in the design of wavelets. The design of the filters is quite different from the classical case ($M = 2$): it is, in general, more difficult, and the wavelets are no longer determined, in an essentially unique manner, by a pair of biorthogonal MRA's.

In [23] Soardi considered spline M-band primal scaling functions of arbitrary degree, and constructed dual scaling functions having arbitrarily high regularity. The spline case is a natural choice, but it is only one possibility out of many others, which could be better suited for specific purposes. In view of possible applications, it is natural to seek a "good" compromise between regularity, support width, vanishing moments and symmetry properties. In the $M = 2$ setting, the spline wavelets are quite popular in digital image processing, but the generalization to the M-band setting proposed in [23] has the drawback of having complicated dual filters and wavelets with large support widths. Other well-known 2-band biorthogonal wavelets are Burt-Adelson wavelets ([5],[10],[18]). They are generated by a one (real) parameter family of symmetric filters, with small support, but enough

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regularity and vanishing moments to make them a good choice in specific applications. The regularity of the dual wavelets (as well as the number of vanishing moments of some primal wavelets) can also be improved by considering extended and maximally smooth dual filters constructed and studied by the author in [18].

In this paper we present a natural M-band generalization of classical Burt-Adelson filters. We are able to construct a one real parameter family of primal and dual scaling functions for any integer $M \geq 2$, and for $M \in \{3,4\}$ we also find simple explicit expression for two different families of wavelets filters. The ones in the first familiy have small supports, while the ones in the second have the interesting properties of symmetry: for $M = 3$ we obtain a symmetric-antisymmetric pair, for $M = 4$ an antisymmetric and two symmetric (with respect to different centers) wavelets. Both the small support width and these symmetry properties are expected to give particularly good results in applications.

Our construction can be compared with the one carried out by Chui and Lian in [6], which lead to 3-band orthonormal symmetric/antisymmetric wavelets with possibly arbitrarily high regularity, and also to two recent constructions of orthonormal families of M-band wavelets by Belogay and Wang [2], and by Bi, Dai and Sun [3]. However, all these constructions were carried out in the orthogonal case, and suffer of some drawbacks of that setting.

In the second part of the paper we investigate the regularity of our scaling functions. The case $M = 2$ has been already fully studied by the author in [18]. Here, for $M = 3$ we are able to determine the critical exponent of almost all our wavelets, and for any $M \geq 3$ we find the Sobolev exponents of all our scaling functions, which leads, in particular, to finding sharp conditions for these scaling functions to generate biorthogonal unconditional systems.

2. Biorthogonal M-band wavelet bases

In this section, we briefly review some basic facts about the construction of M-band biorthogonal, compactly supported, wavelets along the lines of [23].

A family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $\mathbb{L}^2(\mathbb{R})$ is said to be an M-band multiresolution analysis if

- $\bigcup_j V_j$ is dense in $\mathbb{L}^2(\mathbb{R})$ and $\bigcap_j V_j = \{0\}$;
- $V_j \subset V_{j+1}$ and $f(\cdot) \in V_j$ if and only if $f(M\cdot) \in V_{j+1}$;
- there exists $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for V_0.

There exist functions $\{\psi_l : l = 1, \dots, M-1\}$, such that, if we let

$$\psi_{j,k,l}(\cdot) = 2^{j/2}\psi_l(2^j \cdot -k) \ ,$$

the following orthogonality relations are satisfied

$$< \psi_{j,k,l}, \psi_{j',k',l'} >= \delta_{j,j'}\delta_{k,k'}\delta_{l,l'} \ ,$$

and the space W_j, the orthogonal complement of V_j in V_{j+1}, is generated by $\{\psi_{j,k,l}\}_{l=1\dots M-1,k\in\mathbb{Z}}$. Since $\mathbb{L}^2(\mathbb{R}) = \oplus_j W_j$, $\{\psi_{j,k,l}\}_{j,k,l}$ is an orthonormal basis for $\mathbb{L}^2(\mathbb{R})$.

In the biorthogonal setting, we have two M-band MRA's, $\{V_j\}$ and $\{\tilde{V}_j\}$, with associated scaling functions φ, $\tilde{\varphi}$, and wavelets $\{\psi_l\}$, $\{\tilde{\psi}_l\}$, which are biorthogonal in the sense that

$$< \varphi(\cdot - k), \tilde{\varphi}(\cdot - k') >= \delta_{k,k'}$$

and

$$< \psi_{j,k,l}, \tilde{\psi}_{j',k',l'} >= \delta_{j,j'}\delta_{k,k'}\delta_{l,l'}$$

As usual, the multiresolution structure forces the refinability of the scaling functions

$$\begin{aligned} \hat{\varphi}(M\xi) &= m_0(\xi)\varphi(\xi) \\ \hat{\tilde{\varphi}}(M\xi) &= \tilde{m}_0(\xi)\hat{\tilde{\varphi}}(\xi) \end{aligned}$$

and, since $\psi_l \in V_1$, we have the relations

$$\begin{aligned} \hat{\psi}_l(M\xi) &= m_l(\xi)\hat{\psi}_l(\xi) \\ \hat{\tilde{\psi}}_l(M\xi) &= \tilde{m}_l(\xi)\hat{\tilde{\psi}}_l(\xi) \end{aligned}$$

for suitable functions $m_0, m_1, \dots, m_{M-1}$, $\tilde{m}_0, \tilde{m}_1, \dots, \tilde{m}_{M-1}$ in $\mathbb{L}^2(\mathbb{T})$ (called filters). We consider here only $\{m_j\}$,$\{\tilde{m}_j\}$ which are trigonometric polynomials.

One can also start with a M-band subband coding scheme ([27],[14],[15],[26]) with filters

$$m_0, \dots, m_{M-1}, \tilde{m}_0, \dots, \tilde{m}_{M-1} \ ,$$

all trigonometric polynomials. By letting

$$\vartheta_k = \frac{2k\pi}{M} \ ,$$

it is well known that the conditions for perfect reconstruction can be written as

$$\sum_{l=0}^{M-1} m_l(\xi)\overline{\tilde{m}_l(\xi + \vartheta_k)} = \delta_{0,k} \quad \forall k \in \{0, \dots, M-1\} \ . \tag{2.1}$$

These equations imply (see Prop.2 in [23]) the biorthogonality conditions

$$\sum_{k=0}^{M-1} m_{l_1}(\xi + \vartheta_k)\overline{\tilde{m}_{l_2}(\xi + \vartheta_k)} = \delta_{l_1,l_2} \quad \forall l_1, l_2 \in \{0, \dots, M-1\} \ . \tag{2.2}$$

We proceed by first designing the scaling function filters m_0 and $\tilde{m}_0$, satisfying (2.1), i.e.

$$\sum_{k=0}^{M-1} m_0(\xi+\vartheta_k)\overline{\tilde{m}_0(\xi+\vartheta_k)} = 1 \ . \tag{2.3}$$

Once they have been established, one defines, via the usual product formulas,

$$\hat{\varphi}(\xi) = \prod_{k=0}^{\infty} m_0\Big(\frac{\xi}{M^k}\Big) \ , \tag{2.4}$$

$$\hat{\tilde{\varphi}}(\xi) = \prod_{k=0}^{\infty} \tilde{m}_0\Big(\frac{\xi}{M^k}\Big) \ . \tag{2.5}$$

If these scaling functions belong to $\mathbb{L}^2(\mathbb{R})$, we obtain a pair of biorthogonal M-band MRA's. It is a well known fact that families of M-band, compactly supported, biorthogonal wavelets associated with these MRA's exist, and can actually be constructed by means of matrix-based algorithms ([21]). These wavelets have Fourier transform given by

$$\hat{\psi}_l(M\xi) = m_l(\xi)\hat{\varphi}(\xi) \ ,$$
$$\hat{\tilde{\psi}}_l(M\xi) = \tilde{m}_l(\xi)\hat{\tilde{\varphi}}(\xi) \ ,$$

for $l \in \{1,\dots,M-1\}$, where all $m_l, \tilde{m}_l$ are trigonometric polynomials, satisfying the conditions (2.2), and

$$m_l(0) = 0 = \tilde{m}_l(0) \ . \tag{2.6}$$

In order to find an explicit expression for the wavelet filters, one can first look for dual wavelet filters such that

$$Ce^{iM(h+\eta)}m_0(\xi) = \begin{vmatrix} \overline{\tilde{m}_1(\xi+\vartheta_1)} & \dots & \overline{\tilde{m}_{M-1}(\xi+\vartheta_1)} \\ \vdots & \ddots & \vdots \\ \overline{\tilde{m}_1(\xi+\vartheta_{M-1})} & \dots & \overline{\tilde{m}_{M-1}(\xi+\vartheta_{M-1})} \end{vmatrix} \tag{2.7}$$

for some $C \neq 0$ and an integer h, where $\eta = 0$ if $M-1$ is even, and $\eta = 1$ if $M-1$ is odd. After this, in Prop.4 of [23] it is proved that, given $m_0, \tilde{m}_0, \dots, \tilde{m}_{M-1}$ satisfying (2.3) and (2.7), there are unique trigonometric polynomials $m_1, \dots, m_{M-1}$ (obtained by solving the linear system in (2.1)), such that the family

$$m_0, \dots, m_{M-1}, \tilde{m}_0, \dots, \tilde{m}_{M-1} \ ,$$

has perfect reconstruction. The biorthogonality conditions (2.2) then follow, and, if the scaling functions (and hence the wavelets) are in $\mathbb{L}^2(\mathbb{R})$, the fundamental theorem by Cohen, Daubechies and Faveau ([9],[10]), generalized to the case $M > 2$ by Soardi in [23] (and refined by the author in [19]), assures that these filters generate unconditional bases of biorthogonal M-band wavelets.

3. Construction of M-band Burt-Adelson scaling functions

In this section we present the construction, for every $M \geq 2$, of a one-parameter family of biorthogonal symmetric scaling functions with small support, which generalize the 2-band family of Burt-Adelson scaling functions.

We look for scaling function filters, ${}_M m_{0,a}$ and ${}_M \tilde{m}_{0,a}$, depending on a real parameter a,

$$ {}_M m_{0,a}(\xi) = \left(\frac{\sin\left(\frac{M\xi}{2}\right)}{M \sin\left(\frac{\xi}{2}\right)} \right)^2 \Big(a + (1-a)\cos\xi \Big) \tag{3.1} $$

and

$$ \begin{aligned} {}_M \tilde{m}_{0,a}(\xi) = \left(\frac{\sin\left(\frac{M\xi}{2}\right)}{M \sin\left(\frac{\xi}{2}\right)} \right)^2 \Big(1 - b_1(a,M) - b_2(a,M) + \\ b_1(a,M)\cos\xi + b_2(a,M)\cos 2\xi \Big) \ . \end{aligned} \tag{3.2} $$

These filters automatically satisfy the "high and low pass" conditions

$$ \begin{aligned} {}_M m_{0,a}(0) = \quad 1 \quad &= {}_M \tilde{m}_{0,a}(0) \ , \\ {}_M m_{0,a}(\vartheta_k) = \quad 0 \quad &= {}_M \tilde{m}_{0,a}(\vartheta_k) \ , \end{aligned} $$

where, as before, $\vartheta_k = 2k\pi/M$.

The main issue is to prove the existence of a filter ${}_M \tilde{m}_{0,a}$ as above which is dual to ${}_M m_{0,a}$, i.e. it satisfies the biorthogonality relation

$$ \sum_{k=0}^{M-1} {}_M m_{0,a}(\xi + \vartheta_k) \, \overline{{}_M \tilde{m}_{0,a}(\xi + \vartheta_k)} = 1 \ . \tag{3.3} $$

Theorem 3.1. For every $M \geq 2$, and every $a \notin \{0, 1\}$, there exist functions $b_1(a,M), b_2(a,M)$, such that the filters (3.1) and (3.2) satisfy the relation (3.3).

Proof. The filters can be split into a "spline" factor and the residuals

$$ {}_M P_a(\xi) = a + (1-a)\cos\xi \tag{3.4} $$

$$ \begin{aligned} {}_M \tilde{P}_a(\xi) = \ & 1 - b_1(a,M) - b_2(a,M) + b_1(a,M)\cos\xi \\ & + b_2(a,M)\cos 2\xi \ . \end{aligned} \tag{3.5} $$

The choice

$$ b_1(a,M) = -2\frac{a-2}{a-1} b_2(a,M) \ , $$

and simple manipulations yield

$$ \begin{aligned} ({}_M P_a \cdot {}_M \tilde{P}_a)(\xi) = \ & 8 b_2(a,M)(1-a)\cos\xi \sin^4\frac{\xi}{2} \\ & + 2\frac{(a-1)^2 - 2ab_2(a,M)}{a-1}\sin^2\frac{\xi}{2} + 1 \ . \end{aligned} $$

Substituting this in (3.3) gives

$$\begin{aligned}
&\sum_{k=0}^{M-1}\left(\frac{\sin\left(\frac{M(\xi+\vartheta_k)}{2}\right)}{M\sin\left(\frac{\xi+\vartheta_k}{2}\right)}\right)^4 (P_a\tilde{P}_a)(\xi+\vartheta_k)= \\
&\qquad = 8b_2(a,M)(1-a)\,\frac{\sin^4\frac{M\xi}{2}}{M^4}\sum_{k=0}^{M-1}\cos(\xi+\vartheta_k) \\
&\qquad\quad + 2\frac{(a-1)^2-2ab_2(a,M)}{a-1}\,\frac{\sin^2\frac{M\xi}{2}}{M^2}\sum_{k=0}^{M-1}\left(\frac{\sin\left(\frac{M(\xi+\vartheta_k)}{2}\right)}{M\sin\left(\frac{\xi+\vartheta_k}{2}\right)}\right)^2 \\
&\qquad\quad + \sum_{k=0}^{M-1}\left(\frac{\sin\left(\frac{M(\xi+\vartheta_k)}{2}\right)}{M\sin\left(\frac{\xi+\vartheta_k}{2}\right)}\right)^4 \\
&\qquad = \left(1-\beta_M-2\frac{(a-1)^2-2ab_2(a,M)}{2M^2(a-1)}\right)\cos M\xi+\beta_M+ \\
&\qquad\quad + 2\frac{(a-1)^2-2ab_2(a,M)}{2M^2(a-1)}\ ,
\end{aligned}$$

since

$$\sum_{k=0}^{M-1}\cos(\xi+\vartheta_k)=0\ ,$$

$$\sum_{k=0}^{M-1}\left(\frac{\sin\left(\frac{M(\xi+\vartheta_k)}{2}\right)}{M\sin\left(\frac{\xi+\vartheta_k}{2}\right)}\right)^2=1\ ,$$

and

$$\sum_{k=0}^{M-1}\left(\frac{\sin\left(\frac{M(\xi+\vartheta_k)}{2}\right)}{M\sin\left(\frac{\xi+\vartheta_k}{2}\right)}\right)^4=\beta_M+(1-\beta_M)\cos M\xi\ ,$$

for some β_M, depending only on M. We obtain the desired relation (3.3), by choosing $b_2(a,M)$ such that

$$1-\beta_M-\frac{(a-1)^2-2ab_2(a,M)}{M^2(a-1)}=0\ ,$$

i.e.

$$b_2(a,M)=\frac{(a-1)(a+M^2(\beta_M-1)-1)}{2a}\ . \tag{3.6}$$

This is possible (in a unique manner) for every β_M, and for every $a\notin\{0,1\}$. □

In view of (3.6) we let

$$\alpha_M=M^2(1-\beta_M)+1\ , \tag{3.7}$$

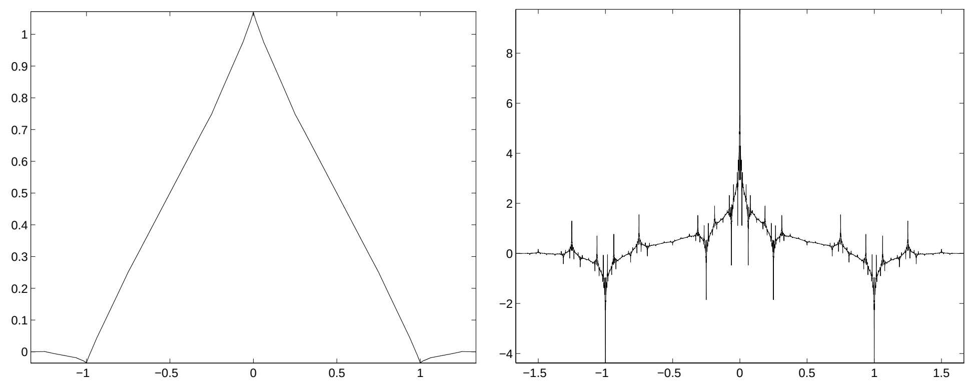

FIGURE 1. ${}_4\varphi_{\frac{6}{5}}$ and ${}_4\tilde{\varphi}_{\frac{6}{5}}$

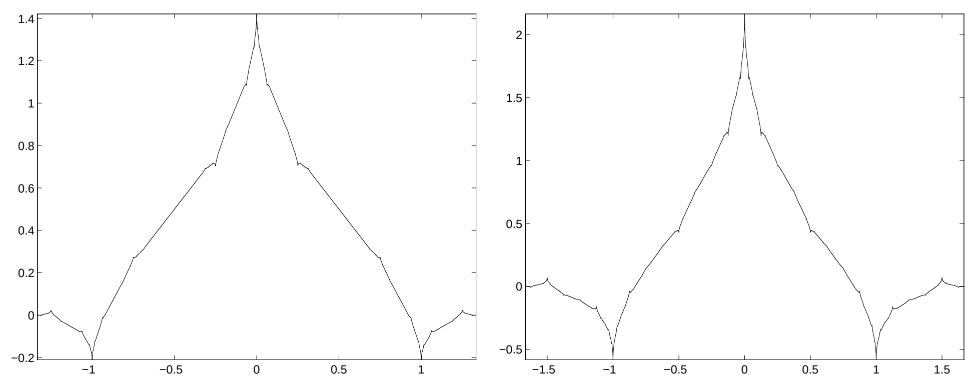

FIGURE 2. ${}_4\varphi_{a^*}$ and ${}_4\tilde{\varphi}_{a^*}$

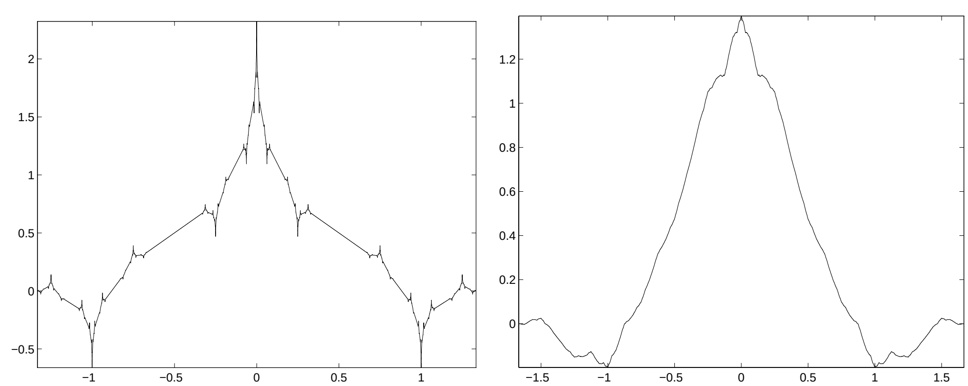

FIGURE 3. ${}_4\varphi_{\frac{14}{5}}$ and ${}_4\tilde{\varphi}_{\frac{14}{5}}$

and in the sequel we will write the residual dual filters (3.6) in the form

$$
\begin{aligned}
{}_M\tilde{P}_a(\xi) = 1 &+ \frac{(a-2)(a-\alpha_M)}{a} - \frac{(a-1)(a-\alpha_M)}{2a} \\
&- \frac{(a-2)(a-\alpha_M)}{a}\cos(\xi) + \frac{(a-1)(a-\alpha_M)}{2a}\cos(2\xi)\ .
\end{aligned}
\tag{3.8}
$$

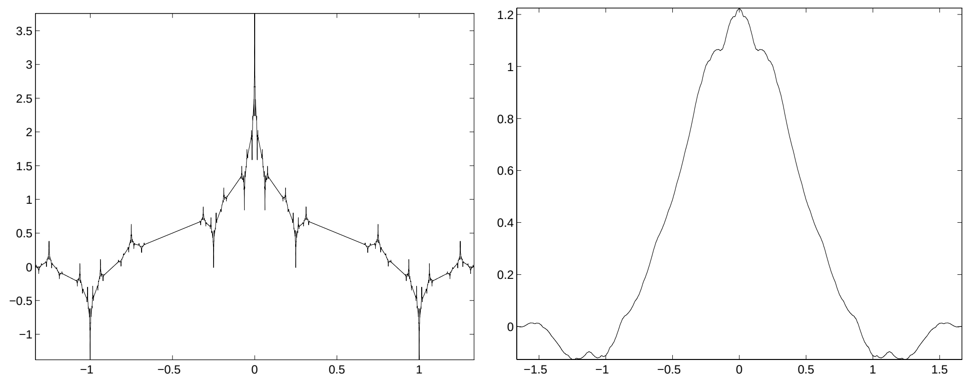

FIGURE 4. ${}_4\varphi_{\frac{7}{2}}$ and ${}_4\tilde{\varphi}_{\frac{7}{2}}$

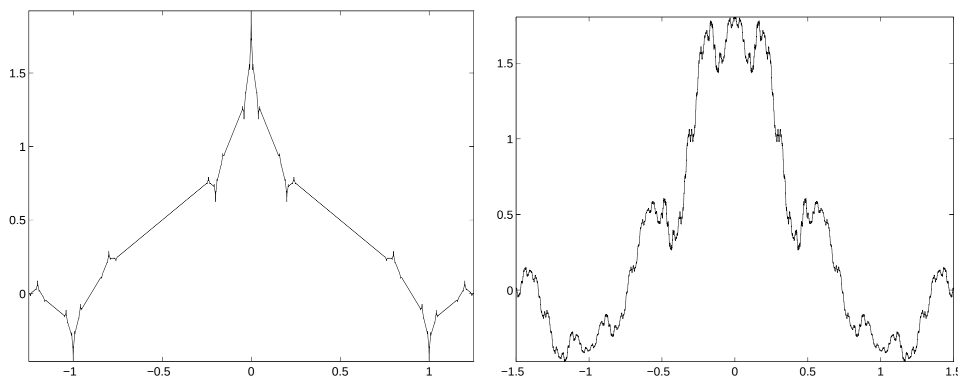

FIGURE 5. ${}_5\varphi_3$ and ${}_5\tilde{\varphi}_3$

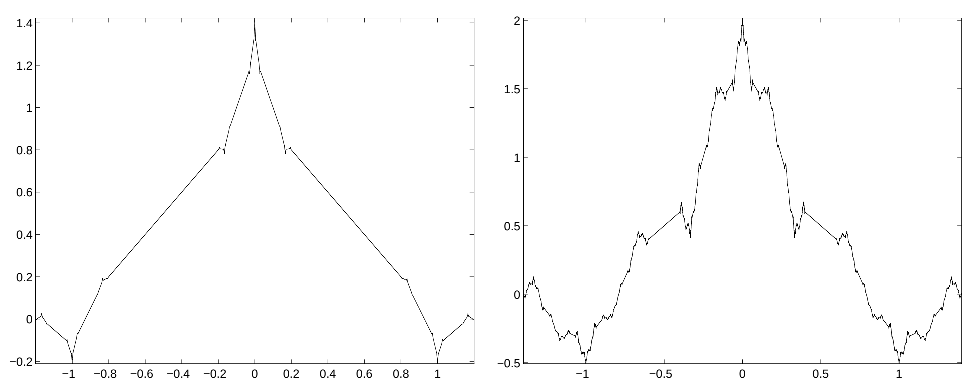

FIGURE 6. ${}_6\varphi_{\frac{5}{2}}$ and ${}_6\tilde{\varphi}_{\frac{5}{2}}$

Remark 3.1. The value of β_M can be found esplicitly in the following way: we write

$$\sum_{k=0}^{M-1}\left(\frac{\sin\left(\frac{M(\xi+\vartheta_k)}{2}\right)}{M\sin\left(\frac{\xi+\vartheta_k}{2}\right)}\right)^4 =$$

$$= \sum_{k=0}^{M-1} e^{i2(M-1)(\xi+\vartheta_k)}\left(\frac{1+e^{-i(\xi+\vartheta_k)}+\cdots+e^{-i(M-1)(\xi+\vartheta_k)}}{M}\right)^4$$

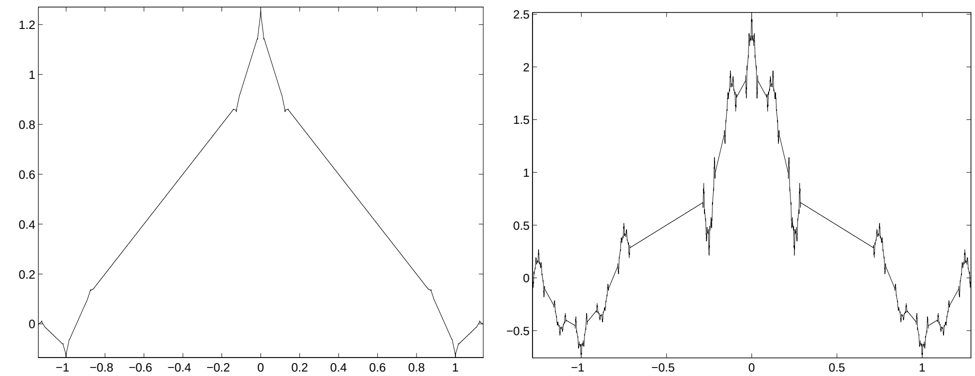

FIGURE 7. ${}_8\varphi_{\frac{5}{2}}$ and ${}_8\tilde{\varphi}_{\frac{5}{2}}$

$$= \frac{1}{M^4}\sum_{k=0}^{M-1} e^{2i(M-1)(\xi+\vartheta_k)} \sum_{l_0+\cdots+l_{M-1}=4} \binom{4}{l_0 \dots l_{M-1}} \prod_{j=1}^{M-1} e^{ijl_j(\xi+\vartheta_k)}$$

$$= \frac{1}{M^4}\sum_{l_0+\cdots+l_{M-1}=4} \binom{4}{l_0 \dots l_{M-1}} \sum_{k=0}^{M-1} e^{i\left(\sum_{j=1}^{M-1} jl_j - 2(M-1)\right)(\xi+\vartheta_k)} .$$

To get β_M, we have to keep the only non-constant term in this last expression, which is obtained when $\sum_{j=1}^{M-1} jl_j = 2(M-1)$:

$$\beta_M = \frac{1}{M^3} \sum_{\substack{l_0+\cdots+l_{M-1}=4 \\ \sum_{j=1}^{M-1} jl_j = 2(M-1)}} \binom{4}{l_0 \dots l_{M-1}} . \tag{3.9}$$

These scaling functions, when in $\mathbb{L}^2(\mathbb{R})$ generate biorthogonal MRA's, and, as explained above, some general theorems guarantee the existence of associated wavelets. In the next two sections we find explicitly two families of wavelets in the case $M \in \{3,4\}$, which have good properties in terms of support width, and symmetry.

Let us observe that

$$\text{supp } \varphi = \left[-\frac{M}{M-1}, \frac{M}{M-1}\right], \quad \text{supp } \tilde{\varphi} = \left[-\frac{M+1}{M-1}, \frac{M+1}{M-1}\right]$$

In particular, for large M the primal and dual scaling functions have similar support width.

We plot in Fig.1-4 some of the 4-band scaling functions corresponding to $a \in \{\frac{6}{5}, a^*, \frac{14}{5}, \frac{7}{2}\}$, where $a^* = \frac{-19+\sqrt{1441}}{10}$ ("almost orthogonal" case).

In Fig.5-7 we plot some examples of our 5,6,8-band scaling functions.

4. Construction of the wavelet filters for M=3

In this section we specialize the construction of the previous section to the case $M = 3$ in order to explicitly construct wavelet families associated with our MRA's.

The dual scaling function filters are

$$
\begin{aligned}
{}_3\tilde{m}_{0,a}(\xi) = \left(\frac{\sin\left(\frac{3\xi}{2}\right)}{3\sin\left(\frac{\xi}{2}\right)}\right)^2 \Big(\frac{3a^2-14a+33}{6a} \\
-\frac{(a-2)(3a-11)}{3a}\cos\xi + \frac{(a-1)(3a-11)}{6a}\cos 2\xi\Big)
\end{aligned} \tag{4.1}
$$

By (2.2), the dual wavelet filters should satisfy (see [23])

$$
C\, e^{3ih\xi}\, {}_3m_{0,a}(\xi) = \begin{vmatrix} \overline{{}_3\tilde{m}_{1,a}(\xi+\vartheta_1)} & \overline{{}_3\tilde{m}_{2,a}(\xi+\vartheta_1)} \\ {}_3\tilde{m}_{1,a}(\xi+\vartheta_2) & {}_3\tilde{m}_{2,a}(\xi+\vartheta_2) \end{vmatrix}, \tag{4.2}
$$

for some constant $C \neq 0$, and an integer h. Since we can write

$$
{}_3m_{0,a}(\xi) = \left(\frac{4}{3}\right)^2 \left(\prod_{k=1}^{2} \cos^2\big(\frac{x}{2}+\frac{k\pi}{3}+\frac{\pi}{2}\big)\right) {}_3P_a(\xi)
$$

we look for ${}_3\tilde{m}_{j,a}$, $j \in \{1,\dots,2\}$, of the form

$$
{}_3\tilde{m}_{j,a}(\xi) = \frac{4}{3}\cos^2\left(\frac{\xi+\pi}{2}\right) {}_3\tilde{P}_{j,a}(\xi)\ ,
$$

so that the first factors make up for the spline term of $m_{0,a}$. Now we have to find ${}_3\tilde{P}_{j,a}$, satisfying

$$
C\, e^{3ih\xi}\, {}_3P_a(\xi) = \begin{vmatrix} \overline{{}_3\tilde{P}_{1,a}(\xi+\vartheta_1)} & \overline{{}_3\tilde{P}_{2,a}(\xi+\vartheta_1)} \\ {}_3\tilde{P}_{1,a}(\xi+\vartheta_2) & {}_3\tilde{P}_{2,a}(\xi+\vartheta_2) \end{vmatrix}. \tag{4.3}
$$

We try with ${}_3\tilde{P}_{j,a}$ having at most three coefficients, in order to minimize the support length, but at the same time allowing symmetric filters. It turns out that even in this case we have many possibilities. The shortest filters are:

$$
{}_3\tilde{P}^I_{1,a}(\xi) = \frac{1-a}{2a} - e^{i\xi}\ , \tag{4.4}
$$

$$
{}_3\tilde{P}^I_{2,a}(\xi) = a\, e^{-i\xi} + \frac{a-1}{2} = -a\, {}_3\tilde{P}^I_{1,a}(\xi)\ . \tag{4.5}
$$

There also exist filters which give a pair of symmetric/antisymmetric wavelets:

$$
{}_3\tilde{P}^{II}_{1,a}(\xi) = \frac{1}{2}(a-1) + a\cos(\xi)\ , \tag{4.6}
$$

$$
{}_3\tilde{P}^{II}_{2,a}(\xi) = 2\sin(\xi)\ . \tag{4.7}
$$

The primal filters are obtained by solving the linear system (2.1). To simplify the notation we write

$$
{}_3m^I_{j,a}(\xi) = \sum_n h^I_{n,j,a} e^{in\xi}\ ,
$$

and list the taps $h^I_{n,j,a}$:

$$\{h^I_{n,1,a}\}_{n=-3\ldots 6} = [\frac{1}{54}\,a^2 - \frac{11}{162}\,a + \frac{4}{81},\ \frac{8}{81} - \frac{1}{27}\,a,\ \frac{16}{81} - \frac{2}{27}\,a,$$
$$\frac{17}{54}\,a - \frac{1}{36}\,a^2 - \frac{11}{108},\ -\frac{5}{54}\,a - \frac{65}{162},$$
$$-\frac{8}{81} - \frac{2}{27}\,a,\ \frac{7}{54}\,a + \frac{1}{54},\ \frac{11}{81} - \frac{1}{27}\,a,$$
$$-\frac{1}{54}\,a + \frac{11}{162},\ \frac{1}{108}\,a^2 - \frac{7}{162}\,a + \frac{11}{324}]\ ,$$

$$\{h^I_{n,2,a}\}_{n=-6\ldots 3} = [-\frac{1}{324}\,\frac{-14\,a + 3\,a^2 + 11}{a},\ \frac{1}{162}\,\frac{-11 + 3\,a}{a},$$
$$\frac{1}{81}\,\frac{-11 + 3\,a}{a},\ -\frac{1}{54}\,\frac{1 + 7\,a}{a},$$
$$\frac{2}{81}\,\frac{3\,a + 4}{a},\ \frac{5}{162}\,\frac{3\,a + 13}{a},$$
$$\frac{1}{108}\,\frac{11 - 34\,a + 3\,a^2}{a},\ \frac{2}{81}\,\frac{-8 + 3\,a}{a},$$
$$\frac{1}{81}\,\frac{-8 + 3\,a}{a},\ -\frac{1}{162}\,\frac{-11\,a + 8 + 3\,a^2}{a}]\ ,$$

and the filters

$$\begin{aligned}{}_3m^{II}_{1,a}(\xi) = -\frac{1}{162a}\Big(&-54\cos(\xi)\,a - 66\cos(\xi)\\ &- 6\cos(5\,\xi)\,a + 22\cos(5\,\xi) + 44\cos(4\,\xi)\\ &- 12\cos(4\,\xi)\,a + 11\cos(6\,\xi)\\ &- 14\cos(6\,\xi)\,a + 3\cos(6\,\xi)\,a^2\\ &- 36\,a\cos(2\,\xi) + 22\cos(3\,\xi)\\ &+ 20\cos(3\,\xi)\,a + 6\cos(3\,\xi)\,a^2 - 33 + 102\,a\\ &- 9\,a^2\Big)\ ,\end{aligned}$$

$$\begin{aligned}{}_3m^{II}_{1,a}(\xi) = &-\frac{97}{162}\sin(\xi) - \frac{1}{54}\sin(\xi)\,a + \frac{11}{162}\sin(5\,\xi)\\ &- \frac{1}{54}\sin(5\,\xi)\,a + \frac{11}{81}\sin(4\,\xi) - \frac{1}{27}\sin(4\,\xi)\,a\\ &+ \frac{1}{108}\sin(6\,\xi)\,a^2 - \frac{7}{162}\sin(6\,\xi)\,a\\ &+ \frac{11}{324}\sin(6\,\xi) - \frac{16}{81}\sin(2\,\xi) - \frac{1}{27}\sin(2\,\xi)\,a\\ &- \frac{5}{162}\sin(3\,\xi) + \frac{16}{81}\sin(3\,\xi)\,a\\ &- \frac{1}{54}\sin(3\,\xi)\,a^2\ .\end{aligned}$$

Observe that ${}_3m^{II}_{1,a}$ is even, while ${}_3m^{II}_{2,a}$ is odd. Hence the primal wavelets enjoy the same symmetry properties as the dual ones. This could have been seen by solving the linear system (2.1) using the Cramer rule, and observing that

$$
\begin{aligned}
{}_3m^{II}_{1,a}(-\xi) &= \begin{vmatrix} \overline{{}_3\tilde m_{0,a}(-\xi+\vartheta_1)} & \overline{{}_3\tilde m^{II}_{2,a}(-\xi+\vartheta_1)} \\ \overline{{}_3\tilde m_{0,a}(-\xi+\vartheta_2)} & \overline{{}_3\tilde m^{II}_{2,a}(-\xi+\vartheta_2)} \end{vmatrix} \\
&= \begin{vmatrix} \overline{{}_3\tilde m_{0,a}(\xi-\vartheta_1)} & \overline{{}_3\tilde m^{II}_{2,a}(\xi-\vartheta_1)} \\ -\overline{{}_3\tilde m_{0,a}(\xi-\vartheta_2)} & -\overline{{}_3\tilde m^{II}_{2,a}(\xi-\vartheta_2)} \end{vmatrix} \qquad (4.8) \\
&= \begin{vmatrix} \overline{{}_3\tilde m_{0,a}(\xi+\vartheta_2)} & \overline{{}_3\tilde m^{II}_{2,a}(\xi+\vartheta_2)} \\ -\overline{{}_3\tilde m_{0,a}(\xi+\vartheta_1)} & -\overline{{}_3\tilde m^{II}_{2,a}(\xi+\vartheta_1)} \end{vmatrix} = {}_3m^{II}_{1,a}(\xi)\ ,
\end{aligned}
$$

where we have used the symmetry of ${}_3\tilde m_{0,a}$, and the antisymmetry of ${}_3\tilde m_{2,a}$, as well. The same trick shows that ${}_3m^{II}_{2,a}$ is antisymmetric.

The supports of the scaling functions and wavelets are as follows:

$$
\begin{array}{ll}
\operatorname{supp}{}_3\varphi_a = \left[-\frac{3}{2},\frac{3}{2}\right]\ , & \operatorname{supp}{}_3\tilde\varphi_a = \left[-2,2\right]\ , \\
\operatorname{supp}{}_3\psi^I_{1,a} = \left[-\frac{5}{2},\frac{3}{2}\right]\ , & \operatorname{supp}{}_3\tilde\psi^I_{1,a} = \left[-\frac{4}{3},1\right]\ , \\
\operatorname{supp}{}_3\psi^I_{2,a} = \left[-\frac{3}{2},\frac{5}{2}\right]\ , & \operatorname{supp}{}_3\tilde\psi^I_{2,a} = \left[-1,\frac{4}{3}\right]\ , \\
\operatorname{supp}{}_3\psi^{II}_{1,a} = \left[-\frac{5}{2},\frac{5}{2}\right]\ , & \operatorname{supp}{}_3\tilde\psi^{II}_{1,a} = \left[-\frac{4}{3},\frac{4}{3}\right]\ , \\
\operatorname{supp}{}_3\psi^{II}_{2,a} = \left[-\frac{5}{2},\frac{5}{2}\right]\ , & \operatorname{supp}{}_3\tilde\psi^{II}_{2,a} = \left[-\frac{4}{3},\frac{4}{3}\right]\ .
\end{array}
$$

5. Construction of the wavelet filters for $M=4$

In this section we construct explicitly two wavelet families in the case $M=4$.

The dual scaling function filters, we constructed in Section 3, are

$$
\begin{aligned}
{}_4\tilde m_{0,a}(\xi) = \left(\frac{\sin\left(\frac{M\xi}{2}\right)}{M\sin\left(\frac{\xi}{2}\right)}\right)^2 \Big(&\frac{a^2-7a+18}{2a} \qquad (5.1) \\
&-\frac{(a-2)(a-6)}{a}\cos\xi+\frac{(a-1)(a-6)}{2a}\cos 2\xi\Big)
\end{aligned}
$$

By reasoning in the same way as in Section 4, we look for ${}_4\tilde m_{j,a}$, $j\in\{1,\dots,3\}$, in the form

$$
{}_4\tilde m_{j,a}(\xi) = \frac{4}{3}\cos^2\left(\frac{\xi+\pi}{2}\right){}_4\tilde P_{j,a}(\xi)\ ,
$$

with ${}_4\tilde P_{j,a}$ satisfying

$$
C\,e^{2ih\xi}\,{}_4P_a(\xi) = \begin{vmatrix} \overline{{}_4\tilde P_{1,a}(\xi+\vartheta_1)} & \overline{{}_4\tilde P_{2,a}(\xi+\vartheta_1)} & \overline{{}_4\tilde P_{3,a}(\xi+\vartheta_1)} \\ \overline{{}_4\tilde P_{1,a}(\xi+\vartheta_2)} & \overline{{}_4\tilde P_{2,a}(\xi+\vartheta_2)} & \overline{{}_4\tilde P_{3,a}(\xi+\vartheta_2)} \\ \overline{{}_4\tilde P_{1,a}(\xi+\vartheta_3)} & \overline{{}_4\tilde P_{2,a}(\xi+\vartheta_3)} & \overline{{}_4\tilde P_{3,a}(\xi+\vartheta_3)} \end{vmatrix}\ . \qquad (5.2)
$$

It turns out that, among various possible choices, we can keep two of the wavelet filters corresponding to the $M=3$ case (see formulas

(4.5) and (4.7)), and add a third wavelet. In this way we obtain

$$ {}_4\tilde{P}^I_{1,a}(\xi) = \frac{1-a}{2a} - e^{i\xi} \ , \tag{5.3} $$

$$ {}_4\tilde{P}^I_{2,a}(\xi) = a\, e^{-i\xi} + \frac{a-1}{2} \ , \tag{5.4} $$

$$ {}_4\tilde{P}^I_{3,a}(\xi) = e^{2i\xi} \ , \tag{5.5} $$

or

$$ {}_4\tilde{P}^{II}_{1,a}(\xi) = \frac{1}{2}(a-1) + a\cos(\xi) \ , \tag{5.6} $$

$$ {}_4\tilde{P}^{II}_{2,a}(\xi) = 2\sin(\xi) \ , \tag{5.7} $$

$$ {}_4\tilde{P}^{II}_{3,a}(\xi) = e^{i2\xi} \ . \tag{5.8} $$

We see that the first two primal filters enjoy exactly the same properties (of length and symmetry) as those in the case $M = 3$, while the third filter is very short (only three coefficients). Proving that (5.2) is satisfied with these choices is a matter of a simple computation.

The primal filters are obtained, as usual, by solving the linear system (2.1). We report here only ${}_4m^{II}_{1,a}$, ${}_4m^{II}_{2,a}$, ${}_4m^{II}_{3,a}$:

$$ \begin{aligned} {}_4m^{II}_{1,a}(\xi) = \frac{3}{512a}\Big(& -12\cos(7\,\xi) + 2\cos(7\,\xi)\,a \\ & + 24\cos(2\,\xi) + 28\cos(2\,\xi)\,a - 6\cos(8\,\xi) + 7\cos(8\,\xi)\,a \\ & - \cos(8\,\xi)\,a^2 + 6\cos(5\,\xi)\,a - 36\cos(5\,\xi) - 91\,a + 5\,a^2 \\ & + 30 - 4\cos(4\,\xi)\,a^2 - 24\cos(4\,\xi) - 12\cos(4\,\xi)\,a \\ & - 12\cos(3\,\xi) + 18\cos(3\,\xi)\,a + 38\,a\cos(\xi) + 60\cos(\xi) \\ & + 4\cos(6\,\xi)\,a - 24\cos(6\,\xi)\Big) \ , \end{aligned} $$

$$ \begin{aligned} {}_4m^{II}_{2,a}(\xi) = & \frac{9}{256}\sin(7\,\xi) - \frac{3}{512}\sin(7\,\xi)\,a \\ & - \frac{39}{128}\sin(2\,\xi) - \frac{3}{256}\sin(2\,\xi)\,a + \frac{3}{1024}\sin(8\,\xi)\,a^2 \\ & + \frac{9}{512}\sin(8\,\xi) - \frac{21}{1024}\sin(8\,\xi)\,a + \frac{27}{256}\sin(5\,\xi) \\ & - \frac{9}{512}\sin(5\,\xi)\,a + \frac{9}{128}\sin(6\,\xi) - \frac{3}{256}\sin(6\,\xi)\,a \\ & - \frac{135}{256}\sin(\xi) - \frac{3}{512}\,a\sin(\xi) - \frac{9}{512}\sin(3\,\xi)\,a - \frac{21}{256}\sin(3\,\xi) \\ & + \frac{3}{256}\sin(4\,\xi) - \frac{3}{512}\sin(4\,\xi)\,a^2 + \frac{57}{512}\sin(4\,\xi)\,a \ , \end{aligned} $$

$$
\begin{aligned}
{}_4m^{II}_{3,a}(\xi) = \frac{27}{64}\Big[&2\,(\frac{1}{36}\,a - \frac{5}{36})\cos(5\,\xi) \\
&+ 2\,(\frac{1}{18}\,a - \frac{5}{18})\cos(4\,\xi) + 2\,(\frac{1}{12}\,a - \frac{1}{72}\,a^2 - \frac{5}{72})\cos(6\,\xi) \\
&+ 2\,(-\frac{5}{12}\,a + \frac{1}{72}\,a^2 - \frac{1}{24})\cos(2\,\xi) + 2\,(\frac{1}{3} + \frac{1}{9}\,a)\cos(\xi) + \frac{1}{9}\,a \\
&+ \frac{11}{9} + 2\,(\frac{1}{12}\,a - \frac{5}{12})\cos(3\,\xi)\Big]e^{2\,i\,\xi}\ .
\end{aligned}
$$

We notice that the filters of the family I (not reported here) have 13 coefficients; on the other side, the filters of family II have $17, 17, 13$ coefficients, two of them are symmetric (with respect to different axes) and one is antisymmetric, exactly as their duals are.

Standard calculations show that the supports of the scaling functions and wavelets are the following:

$$
\begin{array}{ll}
\operatorname{supp}{}_4\varphi_a = \big[-\frac{4}{3},\frac{4}{3}\big]\ , & \operatorname{supp}{}_4\tilde{\varphi}_a = \big[-\frac{5}{3},\frac{5}{3}\big]\ , \\
\operatorname{supp}{}_4\psi^I_{1,a} = \big[-\frac{4}{3},\frac{7}{3}\big]\ , & \operatorname{supp}{}_4\tilde{\psi}^I_{1,a} = \big[-\frac{2}{3},\frac{11}{12}\big]\ , \\
\operatorname{supp}{}_4\psi^I_{2,a} = \big[-\frac{7}{3},\frac{4}{3}\big]\ , & \operatorname{supp}{}_4\tilde{\psi}^I_{2,a} = \big[-\frac{11}{12},\frac{2}{3}\big]\ , \\
\operatorname{supp}{}_4\psi^I_{3,a} = \big[-\frac{1}{6},\frac{7}{6}\big]\ , & \operatorname{supp}{}_4\tilde{\psi}^I_{3,a} = \big[-\frac{4}{3},\frac{4}{3}\big]\ , \\
\operatorname{supp}{}_4\psi^{II}_{1,a} = \big[-\frac{11}{12},\frac{11}{12}\big]\ , & \operatorname{supp}{}_4\tilde{\psi}^{II}_{2,a} = \big[-\frac{4}{3},\frac{4}{3}\big]\ , \\
\operatorname{supp}{}_4\psi^{II}_{2,a} = \big[-\frac{11}{12},\frac{11}{12}\big]\ , & \operatorname{supp}{}_4\tilde{\psi}^{II}_{3,a} = \big[-\frac{4}{3},\frac{4}{3}\big]\ , \\
\operatorname{supp}{}_4\psi^{II}_{3,a} = \big[-\frac{1}{6},\frac{7}{6}\big]\ , & \operatorname{supp}{}_4\tilde{\psi}^{II}_{1,a} = \big[-\frac{4}{3},\frac{4}{3}\big]\ .
\end{array}
$$

6. Critical and Sobolev Exponents of M-band Wavelets

The regularity of scaling functions generated by filters via the infinite product formula (2.5) and, more general, of solutions of refinement equations of the form

$$\varphi(x) = \sum_{k=-K_1}^{K_2} \alpha_k\varphi(Mx-k)\ , \tag{6.1}$$

can be studied on the Fourier transform side by means of the critical and Sobolev exponents b and s_2, respectively. One introduces the trigonometric polynomial

$$m_0(\xi) = \sum_{k=-K_1}^{K_2} \alpha_k e^{-ik\xi}\ , \tag{6.2}$$

and then studies the regularity of the compactly supported distribution

$$\hat{\varphi}(\xi) = \prod_{k=0}^{+\infty} m_0(\xi/M^k)\ , \tag{6.3}$$

in order to establish if it is the Fourier transform of some φ in $\mathbb{L}^1(\mathbb{R})\cap\mathbb{L}^2(\mathbb{R})$, solution of (6.1).

We can always write

$$m_0(\xi) = \Big(\frac{1+e^{-i\xi}+\cdots+e^{-i(M-1)\xi}}{M}\Big)^N \mathcal{L}(\xi)\ , \tag{6.4}$$

for some $N \geq 0$, and a trigonometric polynomial $\mathcal{L}$ such that $\mathcal{L}(\pi) \neq 0$. We will call $\mathcal{L}$ the residual filter. It is clear that the first factor gives a decay of order $|\xi|^{-N}$, for $|\xi| \to \infty$ in the product (6.3) that defines $\hat{\varphi}$. Hence we only need to estimate the growth of $\prod_k \mathcal{L}(\xi/M^k)$.

The critical exponent b is defined by letting

$$b = \inf_j b_j\ , \tag{6.5}$$

where

$$b_j = \log_M \sup_{\xi\in\mathbb{R}} \left[\prod_{k=0}^{j-1} \left|\mathcal{L}(M^k\xi)\right|\right]^{\frac{1}{j}}\ . \tag{6.6}$$

One can prove the sharp pointwise estimate ([7],[11])

$$|\hat{\varphi}(\xi)| \leq C_\epsilon (1+|\xi|)^{-N+b+\epsilon}\ ,$$

for any $\epsilon > 0$, and deduce from it global smoothness properties of φ. Lower bounds for b are easily obtained by considering cycles for the map τ_M on the unit circle $\mathcal{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ in $\mathbb{C}$, defined by $\tau_M(\xi) = M\xi \pmod{2\pi}$. Indeed, let $\gamma = \{\xi_0, \ldots, \xi_{p-1}\}$ be a cycle for τ_M, i.e. $\xi_r = M^r\xi_0$ for $r \in \{1, \ldots, p-1\}$, and $M\xi_{p-1} = \xi_0$. Then (6.5) and (6.6) imply immediately that

$$b_\gamma := \log_M \left[\prod_{k=0}^{p-1} \left|\mathcal{L}(\xi_k)\right|\right]^{\frac{1}{p}} \leq b \leq \sup_\xi |\mathcal{L}(\xi)|\ . \tag{6.7}$$

These simple estimates have proven to be very effective. For example, the equality $b = b_\gamma$, $\gamma = \{-\frac{2\pi}{3}, \frac{2\pi}{3}\}$, holds for all Daubechies compactly supported wavelets ([11], [29], [7], [8]), and for classical Burt-Adelson wavelets the critical exponent is always given by $\max\{b_{\{0\}}, b_{\{-2\pi/3, 2\pi/3\}}\}$ ([18]).

The Sobolev exponent is defined as

$$s_2(\varphi) = \sup\{s : \varphi \in \mathcal{H}^s\}\ ,$$

where as usual

$$\mathcal{H}^s = \Big\{ f : ||f||^2_{\mathcal{H}^s} = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s d\xi < \infty \Big\}\ .$$

This exponent gives more precise estimates for the regularity of φ, for example in Hölder sense, and of course it also allows to determine exactly whether a function is in $\mathbb{L}^2(\mathbb{R})$. It is well known that s_2 can be determined by studying the transition operator

$$\begin{aligned} {}_M T_P : \mathcal{C}([0,1]) &\to \mathcal{C}([0,1]) \\ f &\mapsto \sum_{k=0}^{M-1} P(\xi/M + \vartheta_k) f(\xi/M + \vartheta_k) \ , \end{aligned}$$

where $P = |\mathcal{L}|^2$. In fact, the spectral radius ρ of T_P is related to $s_2(\varphi)$ by the formula

$$s_2(\varphi) = N - \frac{1}{2}\log_M \rho$$

where N is as in (6.4) ([11],[12],[13],[16]) . A most important fact is that the spectral radius of T_P is the same as that of the restriction of the operator to certain invariant finite-dimensional subspaces. More precisely, if P is a cosine polynomial of degree L, the distinguished subspace

$$\mathcal{F}_L = \left\{ \sum_{k=0}^{L} \gamma_k \cos(k\xi) : (\gamma_k)_k \in \mathbb{C}^{L+1} \right\} \tag{6.8}$$

is invariant under the action of T_P, and ([12],[13])

$$\rho\big(T_P\big|_{\mathcal{F}_L}\big) = \rho(T_P) \quad ,$$

so that the problem is reduced to finding the greatest eigenvalue of $T_P\big|_{\mathcal{F}_L}$. This is the technique we will use in the following sections to determine s_2 for all our scaling functions.

Finally, we recall the (sharp) estimate (see [28],[7])

$$N - b - \frac{1}{2} \le s_2 \le N - b \ .$$

6.1. **Critical Exponent of short 3-band filters.** In this section we will always consider filters in the form

$$m_0(\xi) = \left(\frac{\sin\left(\frac{3\xi}{2}\right)}{3\sin\left(\frac{\xi}{2}\right)} \right)^{L+1} \mathcal{L}(\xi) \ , \tag{6.9}$$

with $m_0(0) = 1$, and

$$\mathcal{L}(\xi) = \sum_{m=0}^{3} a_m \cos(m\xi) \ , \tag{6.10}$$

such that $\mathcal{L}(\xi) \ge 0$ and $\mathcal{L}(\pi) \ne 0$. We want to find the critical exponents of these filters. The main result is the following

Theorem 6.1. Suppose m_0 and $\mathcal{L}$ are as above. Let b be the critical exponent of m_0. If $a_2 \ge 0$

$$b = \begin{cases} \log_3 \mathcal{L}(\pi) & \text{if } a_1 + a_3 \le 0 \\ 0 & \text{if } a_1 + a_3 \ge 0 \end{cases} . \tag{6.11}$$

If $a_2 \le 0$ and one of the following conditions is satisfied:

- $2a_2 \le a_1 + a_3 \le 0$,
- $-2a_2 \ge a_1 + a_3 \ge 0$,

then

$$b = \log_3 \mathcal{L}\left(\frac{\pi}{2}\right) .$$

Remark 6.1. In the notation of (6.7), Theorem (6.1) says that, when any of the above hyphoteses are satisfied,

$$b = \max\{b_{\{0\}}, b_{\{\pi\}}, b_{\{\pi/2,3\pi/2\}}\} .$$

Proof. If $a_2 \ge 0$, we have the following chain of inequalities:

$$\begin{aligned}
&\left(\prod_{k=0}^{j-1} \mathcal{L}(3^k\xi)\right)^{\frac{1}{j}} \\
&\qquad \le a_0 + a_1 \frac{1}{j}\sum_{k=0}^{j-1} \cos(3^k\xi) + a_2 \frac{1}{j}\sum_{k=0}^{j-1} \cos(3^k 2\xi) + a_3 \frac{1}{j}\sum_{k=0}^{j-1} \cos(3^{k+1}\xi) \\
&\qquad \le a_0 + (a_1 + a_3)\frac{1}{j}\sum_{k=0}^{j-1} \cos(3^k\xi) + a_2 + \mathcal{O}(1/j) \\
&\qquad \le \mathcal{O}(1/j) + \begin{cases} a_0 + a_1 + a_2 + a_3 = \mathcal{L}(0) & \text{if } a_1 + a_3 \ge 0 \\ a_0 - a_1 + a_2 - a_3 = \mathcal{L}(\pi) & \text{if } a_1 + a_3 \le 0 \end{cases} ,
\end{aligned}$$

with $\mathcal{O}(1/j)$ uniform in ξ. Passing to the sup on both sides, then letting j to infinity, we see that this inequality, together with (6.7), implies the first part of the theorem.

When $a_2 \le 0$,

$$\begin{aligned}
&\left(\prod_{k=0}^{j-1} \mathcal{L}(3^k\xi)\right)^{\frac{1}{j}} \\
&\le a_0 + (a_1 + a_3)\frac{1}{j}\sum_{k=0}^{j-1} \cos(3^k\xi) - a_2 + 2a_2 \frac{1}{j}\sum_{k=0}^{j-1} \cos^2(3^k\xi) + \mathcal{O}(1/j) \\
&\le \mathcal{O}(1/j) \\
&+ \begin{cases} \mathcal{L}\left(\frac{\pi}{2}\right) - 2a_2 \frac{1}{j}\sum_{k=0}^{j-1} \left(\cos(3^k\xi) - \cos^2(3^k\xi)\right) & \text{if } -2a_2 \ge a_1 + a_3 \ge 0 \\ \mathcal{L}\left(\frac{\pi}{2}\right) + 2a_2 \frac{1}{j}\sum_{k=0}^{j-1} \left(\cos(3^k\xi) + \cos^2(3^k\xi)\right) & \text{if } 2a_2 \le a_1 + a_3 \le 0 \end{cases}
\end{aligned}$$

The thesis follows as above, if we show that

$$\lim_{j\to\infty} \sup_{\mathbb{R}} \frac{1}{j}\sum_{k=0}^{j-1} \left(\cos(3^k\xi) - \cos^2(3^k\xi)\right) = 0 , \tag{6.12}$$

$$\lim_{j\to\infty} \inf_{\mathbb{R}} \frac{1}{j}\sum_{k=0}^{j-1} \left(\cos(3^k\xi) + \cos^2(3^k\xi)\right) = 0 . \tag{6.13}$$

In order to prove (6.13), we solve for $\cos(2\xi)$ the identity

$$|1+e^{i\xi}+e^{2i\xi}+e^{3i\xi}|^2 = 4+6\cos(\xi)+4\cos(2\xi)+2\cos(3\xi)\ ,$$

and substitute:

$$\begin{aligned}
&\frac{1}{j}\sum_{k=0}^{j-1}\left(\cos(3^k\xi)+\cos^2(3^k\xi)\right)\\
&\qquad = \frac{1}{j}\sum_{k=0}^{j-1}\left(\cos(3^k\xi)+1/2+1/2\cos(3^k 2\xi)\right)\\
&\qquad = \frac{1}{j}\sum_{k=0}^{j-1}\cos(3^k\xi)+\frac{1}{2}+\frac{1}{8j}\sum_{k=0}^{j-1}|1+e^{3^k i\xi}+e^{3^k 2i\xi}+e^{3^{k+1}i\xi}|^2\\
&\qquad\quad -1/2-\frac{3}{4j}\sum_{k=0}^{j-1}\cos(3^k\xi)-\frac{1}{4j}\sum_{k=1}^{j}\cos(3^k\xi)\\
&\qquad = \frac{1}{8j}\sum_{k=0}^{j-1}|1+e^{3^k i\xi}+e^{3^k 2i\xi}+e^{3^{k+1}i\xi}|^2+\mathcal{O}(1/j)
\end{aligned}$$

with $\mathcal{O}(1/j)$ uniform in ξ. Taking the inf of the last term leaves a $\mathcal{O}(1/j)$, which goes to 0 as j goes to infinity, uniformly in ξ. One can prove (6.12) in a completely analogous manner, but starting with

$$|1-e^{i\xi}+e^{2i\xi}-e^{3i\xi}|^2 = 4-6\cos(\xi)+4\cos(2\xi)-2\cos(3\xi)\ .$$

□

Remark 6.2. Theorem (6.1) is not exhaustive: as we shall see when we shall apply this result in the study of the critical exponent of our 3-band scaling functions, there exists nonnegative residual filters which do not satisfy any of the hypotheses.

6.2. **Regularity of Burt-Adelson 3 band wavelets.** We apply the results of the previous section to determine the critical exponent of the new wavelets we have constructed. We also study the Sobolev exponents.

Theorem 6.2. The critical exponents b_a and $\tilde{b}_a$ of the primal and dual scaling functions generated by the filters ${}_3m_{0,a}$ and ${}_3\tilde{m}_{0,a}$, defined in (3.1) and (4.1), respectively, are

$$b_a = \begin{cases} 0 & \text{if } \frac{1}{2}\le a\le 1 \\ \log_3(2a-1) & \text{if } a\ge 1 \end{cases}\ ,$$

$$\tilde{b}_a = \begin{cases} \log_3\frac{6a^2-31a+44}{3a} & \text{if } 0<a\le\frac{4}{3} \text{ or } a\ge\frac{11}{3} \\ \log_3\frac{11}{3a} & \text{if } \frac{3}{2}\le a\le\frac{11}{3} \end{cases}\ .$$

The Sobolev exponents s_a and $\tilde{s}_a$ of the primal and dual scaling functions are

$$\begin{aligned} s_a =& 2 - \frac{1}{2}\log_3 \frac{\frac{9}{2}a^2 - 14a + 33}{6a} \\ \tilde{s}_a =& \frac{1}{a^2}\Big[\frac{69}{32}a^4 - \frac{179}{8}a^3 + \frac{4363}{48}a^2 - \frac{4015}{24}a + \frac{3993}{32} + \frac{1}{96}\big(77084865 \\ & - 221467752a + 292427476a^2 - 22463912a^2 + 1016141438a^2 \\ & - 32506872a^5 + 6243588a^6 - 687096a^7 + 33129a^8\big)^{\frac{1}{2}}\Big] \ . \end{aligned}$$

For

$$a \in \mathcal{I}_3^{adm} := \left(0.8673947716, \frac{1+4\sqrt{10}}{3}\right) ,$$

the scaling functions ${}_3\varphi_a$, ${}_3\tilde{\varphi}_a$ generated by ${}_3m_{0,a}$ and ${}_3\tilde{m}_{0,a}$ are in $\mathbb{L}^2(\mathbb{R})$ and give rise to biorthogonal MRA's. For these values of a, any biorthogonal wavelet filters which are trigonometric polynomials generate unconditional biorthogonal 3-band wavelets bases for $\mathbb{L}^2(\mathbb{R})$.

Proof. Let us recall the expressions of the residuals ${}_3P_{0,a}$ and ${}_3\tilde{P}_{0,a}$

$$\begin{aligned} {}_3P_{0,a}(\xi) &= a + (1-a)\cos\xi \\ {}_3\tilde{P}_{0,a}(\xi) &= \frac{3a^2 - 14a + 33}{6a} - \frac{(a-2)(3a-11)}{3a}\cos\xi + \\ &\quad + \frac{(a-1)(3a-11)}{6a}\cos 2\xi \ . \end{aligned}$$

Since ${}_3P_{0,a}$ is even and monotone on $[0,\pi]$, its maximum is attained at 0 or at π. On the other hand, $\{\pi\}$ is a cycle for τ_M, for any odd M, so estimate (6.7) (with $\mathcal{L} =_3 P_{0,a}$ gives b_a as desired. Observe that the same argument actually holds for any odd M (and asymptotically for even M, by considering the cycle $\gamma = \{\frac{M\pi}{M+1}, -\frac{M\pi}{M+1}\}$).

To study the positiveness of the dual residuals, we calculate

$$\frac{d}{d\xi}{}_3\tilde{P}_{0,a} = -\frac{3a-11}{3a}\sin\xi\big(2(a-1)\cos\xi + 2 - a\big) \ .$$

We have flat points at $0,\pi$ and, for $a \in (-\infty, 0) \cup (\frac{4}{3}, \infty)$, another local extrema at

$$\overline{\xi}_a = \arccos\left(\frac{a-2}{2(a-1)}\right) .$$

A simple computation yields

$${}_3\tilde{P}_{0,a}(\overline{\xi}_a) = \frac{3a^2 - 23a + 12}{12(1-a)} \ ,$$

which is nonnegative for

$$a \in \left[\frac{4}{3}, \frac{23+\sqrt{385}}{6}\right] .$$

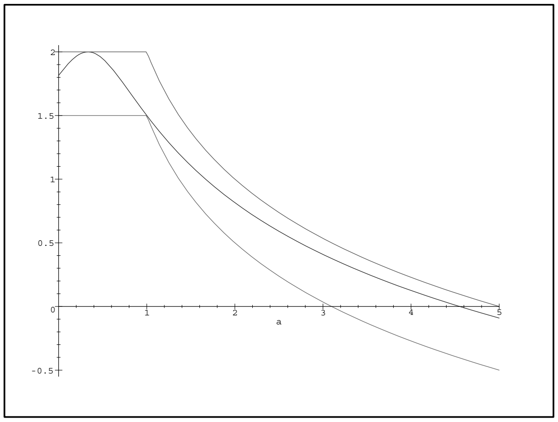

Figure 8. Regularity of 3-band Burt-Adelson primal scaling functions, together with the estimates $N - b_a - 1/2 \le s_2 \le N - b_a$

Since

$$_3\tilde{P}_{0,a}(\pi) = \frac{6a^2 - 31a + 44}{3a} \ge 0 \quad \text{iff} \quad a \in (0, \infty) \ ,$$

we deduce that $_3\tilde{P}_{0,a}(\xi) \ge 0$ for all ξ if and only if

$$a \in \tilde{\mathcal{I}}_P := \left(0, \frac{23 + \sqrt{385}}{6}\right] \ .$$

We apply Theorem 6.1 to determine the critical exponents for the dual residuals. Following the notation of theorem (6.1), we let

$$a_1(a) := -\frac{(a-2)(3a-11)}{3a} \ , \quad a_2(a) := \frac{(a-1)(3a-11)}{6a} \ .$$

Simple computations yield

$$\begin{aligned}
a_1(a) \ge 0 \quad & \text{iff} \quad a \in \big(-\infty, 0\big) \cup \left[2, \frac{11}{3}\right] \ , \\
a_2(a) \ge 0 \quad & \text{iff} \quad a \in \big(0, 1\big] \cup \left[\frac{11}{3}, \infty\right) \ , \\
2a_2(a) \le a_1(a) \le 0 \quad & \text{iff} \quad a \in \left[\frac{3}{2}, 2\right] \cup \left\{\frac{11}{3}\right\} \ , \\
-2a_2(a) \ge a_1(a) \ge 0 \quad & \text{iff} \quad a \in \left[2, \frac{11}{3}\right] \ .
\end{aligned}$$

These relations and the restriction $a \in \tilde{\mathcal{I}}_P$, allow an application of theorem (6.1) for

$$a \in \big(0, 1\big] \cup \left[\frac{3}{2}, \infty\right) \ .$$

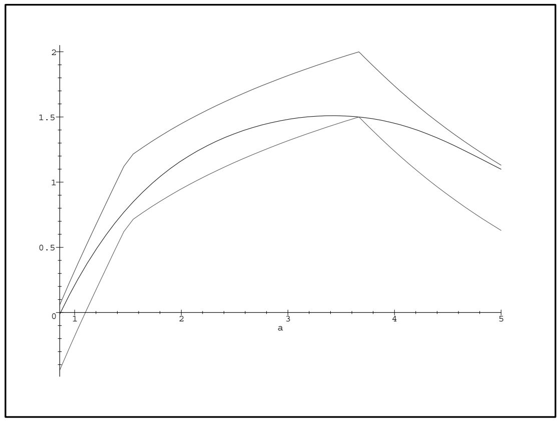

FIGURE 9. Regularity of 3-band Burt-Adelson dual scaling functions: $N - \tilde{b}_a - 1/2 \leq s_2 \leq N - \tilde{b}$. The critical exponent b_a has been plot on $(4/3, 3/2)$ according to Conjecture 1.

For $a \in \left[1, \frac{4}{3}\right]$ it is easily shown that

$$\sup_{\xi} \tilde{P}_{0,a}(\xi) = \tilde{P}_{0,a}(\pi) \ ,$$

and hence $\tilde{b}_a = \log_3 P_{0,a}(\pi)$ for these values of a.

The continuity of the critical exponent and the guess that, at least for these filters of low degree, the relation $b = b_\gamma$ holds for some short cycle γ, leads to the following

Conjecture 1. We have

$$\tilde{b}_a = \log_3 \tilde{P}_{0,a}(\pi) \quad \text{for} \quad a \in \left[\frac{4}{3}, \frac{3}{2}\right] \ .$$

The Sobolev exponent can be found by studying the spectral radii of the transition operators ${}_3T_{|P_a|^2}$ and ${}_3T_{|\tilde{P}_a|^2}$ associated respectively to the primal and dual MRA's. As explained before, we can consider the restrictions ${}_3T_{|P_a|^2}\big|_{\mathcal{F}_2}$ and ${}_3T_{|\tilde{P}_a|^2}\big|_{\mathcal{F}_4}$. Since

$$|P_a|^2(\xi) = a^2 + 2a(1-a)\cos\xi + (1-a)^2\cos^2\xi \ ,$$

a matrix representation of ${}_3T_{|P_a|^2}\big|_{\mathcal{F}_2}$, with respect to the bases $\{1, \cos(\cdot), \cos^2(\cdot)\}$ and $\{1, \cos(\cdot), \cos(2\cdot)\}$, is the following

$$\begin{pmatrix} \frac{9}{2}a^2 - 3a + \frac{3}{2} & 0 & 0 \\ -3a^2 + 3a & \frac{3}{4}a^2 - \frac{3}{2} + \frac{3}{4}a & 0 \\ \frac{3}{4}a^2 - \frac{3}{2}a + \frac{3}{4} & -3a^2 + 3a & 0 \end{pmatrix} \ .$$

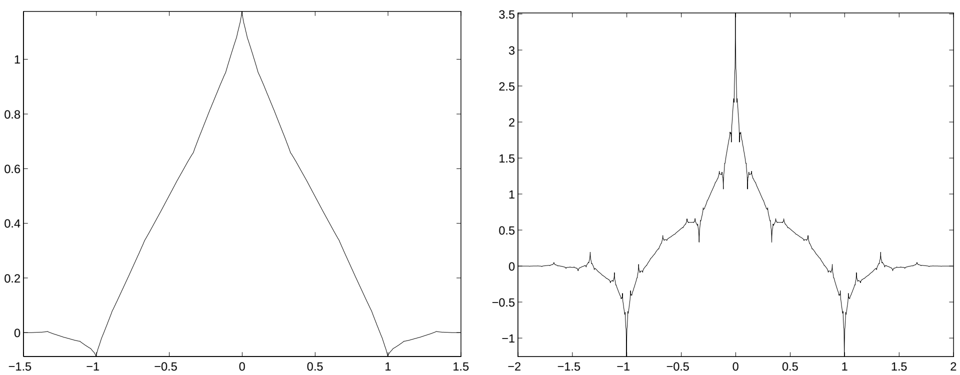

FIGURE 10. ${}_3\varphi_{\frac{13}{10}}$ and ${}_3\tilde{\varphi}_{\frac{13}{10}}$

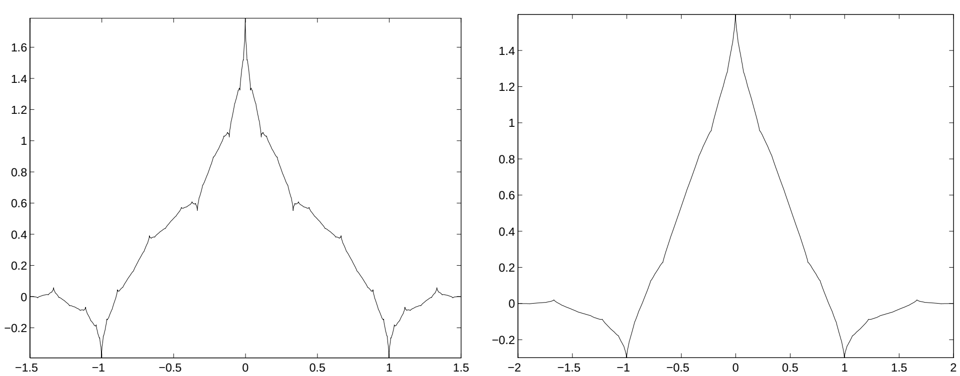

FIGURE 11. ${}_3\varphi_{\frac{\sqrt{33}}{3}}$ and ${}_3\tilde{\varphi}_{\frac{\sqrt{33}}{3}}$

From this one easily deduces that the spectral radius of ${}_3T_{|P_a|^2}\big|_{\mathcal{F}_2}$, and hence of ${}_3T_{|P_a|^2}$, is

$$\rho\big({}_3T_{|P_a|^2}\big) = \frac{9}{2}a^2 - 3a + \frac{3}{2}\ .$$

The calculations for the matrix representing ${}_3T_{|\widetilde{P}_a|^2}\big|_{\mathcal{F}_4}$ are longer, but it is perhaps worth noticing that the determination of the eigenvalues leads to an equation of order three (and not five as one would expect): we will see later that also for $M > 3$ the kernel of ${}_3T_{|\widetilde{P}_a|^2}\big|_{\mathcal{F}_4}$ is non-trivial. The characteristic equation can thus be solved algebraically, leading to the Sobolev exponent of the dual scaling functions.

Finally, to get the interval of admissibility $\mathcal{I}_3^{adm}$ it is sufficient, by Cohen, Daubechies and Faveau's Theorem ([9],[10]) (or, more precisely, by its M-band generalizations in [19] and [23]) to solve the system $\{s_2 > 0, \tilde{s}_2 > 0\}$. The first inequality is satisfied for

$$a \in \Big(\frac{1-4\sqrt{10}}{3}, \frac{1+4\sqrt{10}}{3}\Big)\ ,$$

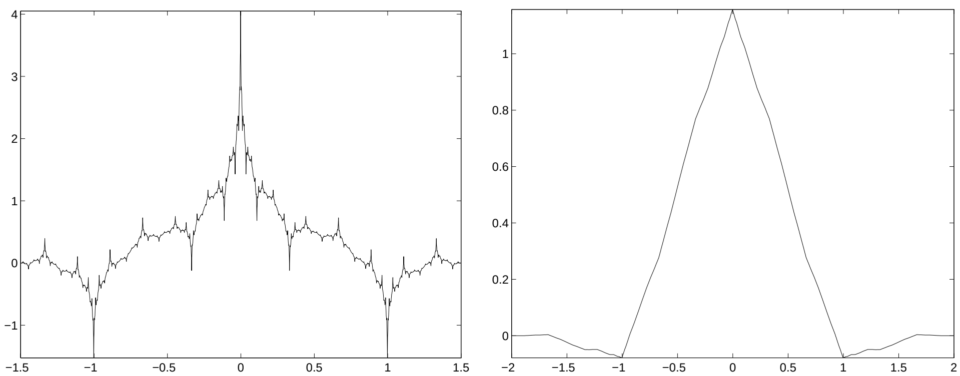

FIGURE 12. ${}_3\varphi_{\frac{14}{5}}$ and ${}_3\tilde{\varphi}_{\frac{14}{5}}$

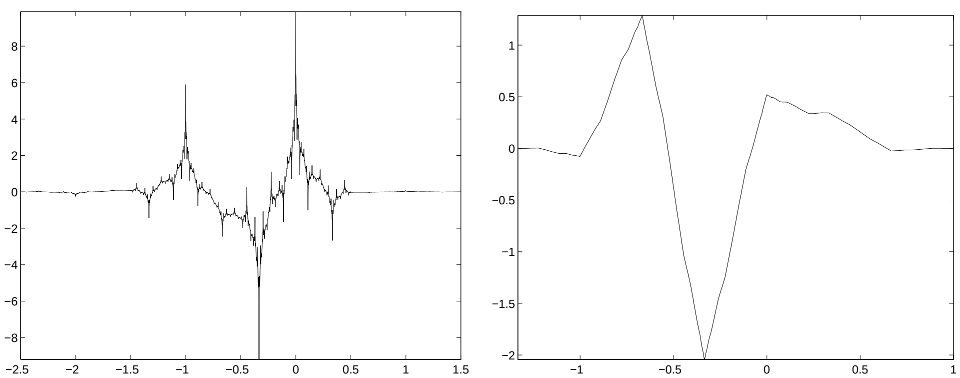

FIGURE 13. ${}_3\psi^I_{1,\frac{14}{5}}$ and ${}_3\tilde{\psi}^I_{1,\frac{14}{5}}$

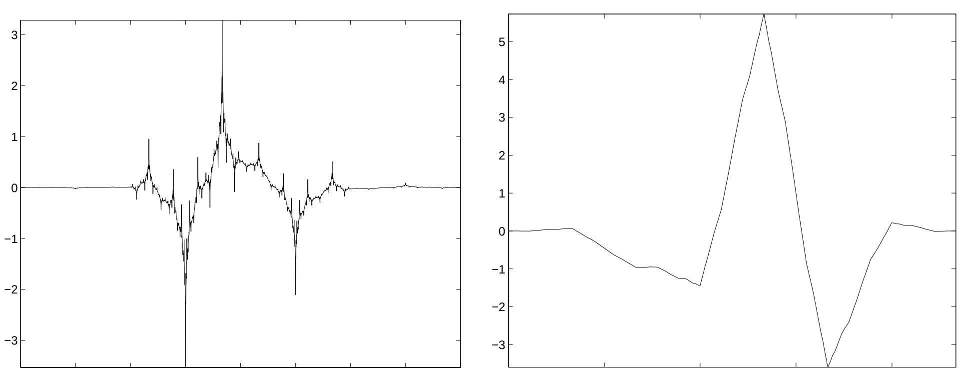

FIGURE 14. ${}_3\psi^I_{2,\frac{14}{5}}$ and ${}_3\tilde{\psi}^I_{2,\frac{14}{5}}$

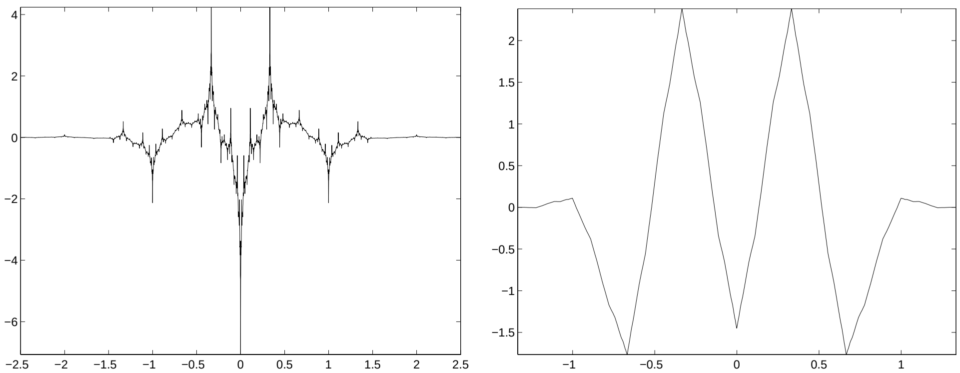

FIGURE 15. ${}_3\psi^{II}_{1,\frac{14}{5}}$ and ${}_3\tilde{\psi}^{II}_{1,\frac{14}{5}}$

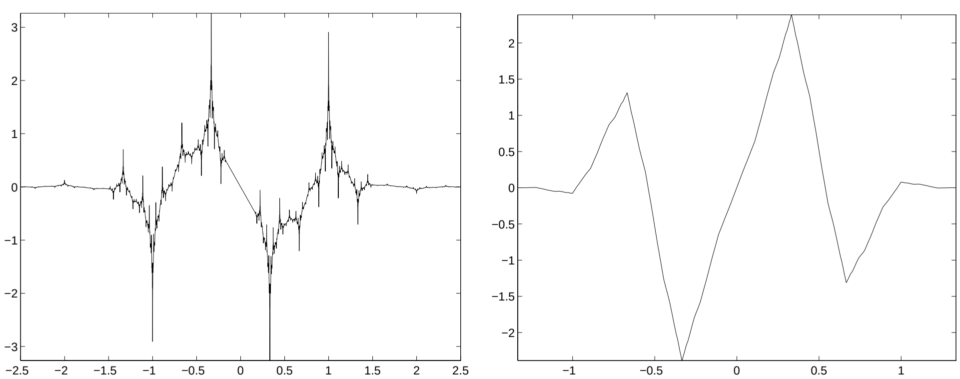

FIGURE 16. ${}_3\psi^{II}_{2,\frac{14}{5}}$ and ${}_3\tilde{\psi}^{II}_{2,\frac{14}{5}}$

and the second one for

$$a \in (0.8673947716, 8.831599978)\ .$$

The last assertion of the theorem then follows by taking the intersection of these two intervals of admissibility. □

We plot in Fig.10-16 the scaling functions corresponding to $a \in \{\frac{13}{10}, \frac{\sqrt{33}}{3}, \frac{14}{5}\}$ and the wavelets only for $a = \frac{14}{5}$, since the value of a doesn't affect the "qualitative structure" of the wavelets. The value $a = \frac{\sqrt{33}}{3}$ minimizes the $\mathbb{L}^2(\mathbb{T})$ distance between m_0 and $\tilde{m}_0$, and the corresponding biorthogonal bases are expected to be nearly orthogonal. This could be a good choice of wavelets for digital image processing, as is remarked, in the $M = 2$ case, in [1].

6.3. **Sobolev Exponent of M-band Burt-Adelson scaling function, $M > 3$.** Since our residual filters are rather short, and their length does not depend on M, it is possible to find explicit algebraic expressions for the spectral radii of the associated transition operators.

6.3.1. *Regularity of the primal filters.* We have

$$|P_a|^2(\xi) = a^2 + 2a(1-a)\cos\xi + (1-a)^2\cos^2\xi \ ,$$

so we can consider the restriction ${}_MT_{|P_a|^2}\big|_{\mathcal{F}_2}$. For $M \geq 4$, a matrix representation of this restricted operator (with respect to the bases $\{1, \cos(\cdots), \cos^2(\cdot)\}$ and $\{1, \cos(\cdots), \cos(2\cdot)\}$) is the following:

$$\begin{pmatrix} \frac{3}{2}Ma^2 - aM + \frac{M}{2} & 0 & 0 \\ Ma(1-a) & 0 & 0 \\ \frac{11}{8}Ma^2 - \frac{3}{4}Ma + \frac{M}{8} & c & 0 \end{pmatrix} \ , \tag{6.14}$$

where c is different from zero only when $M = 4$. Here we have used the simple equalities

$$\sum_{k=0}^{M-1} \cos^2(\xi + \vartheta_k) = \frac{M}{2} \ ,$$
$$\sum_{k=0}^{M-1} \cos^3(\xi + \vartheta_k) = 0 \quad \text{if } M > 3 \ ,$$
$$\sum_{k=0}^{M-1} \cos^4(\xi + \vartheta_k) = \frac{3}{8}M \ ,$$

which can be derived by writing

$$\begin{aligned} \sum_{k=0}^{M-1} \cos^l(\xi + \vartheta_k) &= \frac{1}{2^l} \sum_{k=0}^{M-1} \sum_{j=0}^{l} \binom{l}{j} e^{ij\vartheta_k} e^{-i(l-j)\vartheta_k} \\ &= \frac{1}{2^l} \sum_{j=0}^{l} \binom{l}{j} \sum_{k=0}^{M-1} e^{i(2j-l)\vartheta_k} \ . \end{aligned}$$

We immediately deduce that

$$\rho\big({}_MT_{|P_a|^2}\big|_{\mathcal{F}_2}\big) = \frac{3}{2}Ma^2 - Ma + \frac{M}{2} \ , \tag{6.15}$$

for $M > 3$. From the relation

$$s_2 = 2 - \frac{1}{2}\log_M \rho\big({}_MT_{|P_a|^2}\big|_{\mathcal{F}_2}\big) \ ,$$

we deduce that the admissibility inequality $s_2 > 0$ is satisfied if and only if

$$a \in M\Big(\frac{1}{3} - \sqrt{\frac{M + 3 \cdot 2^{2M+1}}{9M}}, \frac{1}{3} + \sqrt{\frac{M + 3 \cdot 2^{2M+1}}{9M}}\Big) \ . \tag{6.16}$$

6.3.2. *Regularity of the dual filters.* The study of the spectral radius of the transition operators associated with the dual filters is more involved. For $M \geq 6$ we will find a neat expression for it, but when $M \in \{4,5\}$, there seems no way to avoid some calculations. We stress the fact that all of them can be carried out algebraically, and present briefly their results.

Recall that since $|_M\tilde{P}_a|^2$ is a cosine polynomial of degree 4, we can study the restrictions ${}_MT_{|_M\tilde{P}_a|^2}\big|_{\mathcal{F}_4}$. The matrix representing ${}_4T_{|_4\tilde{P}_a|^2}\big|_{\mathcal{F}_4}$ on the bases $\{1,\cos(\cdot),\cos^2(\cdot)\}$ and $\{1,\cos(\cdot),\cos(2\cdot)\}$ is almost lower triangular, and the spectral radius is

$$\begin{aligned}\rho\big({}_4T_{|_4\tilde{P}_a|^2}\big|_{\mathcal{F}_4}\big) =&\frac{1}{a^2}\Big[-273a^2+\frac{435}{4}a^2+279+\frac{5}{4}a^4-19a^3+\frac{1}{4}\big(779233a^4\\ &-2228664a^3+1826064+57a^8-1752a^7+23182a^6\\ &-171760a^5-4040928a+3965112a^2\big)^{\frac{1}{2}}\Big]\,,\end{aligned}$$

so that the interval of admissibility for both φ_a and $\tilde{\varphi}_a$ is

$$\mathcal{I}_4^{adm} := \left(0.9165579310, \frac{1+\sqrt{385}}{3}\right)\,.$$

When $M=5$, similar computations lead to

$$\rho\big({}_5T_{|_5\tilde{P}_a|^2}\big|_{\mathcal{F}_4}\big) = \frac{5}{8}\frac{2835-2844a+1054a^2-148a^3+7a^4}{a^2}\,,$$

from where we can deduce the interval of admissibility

$$\mathcal{I}_5^{adm} = \left(.9694411335, \frac{5+\sqrt{30745}}{15}\right)\,.$$

Finally, for $M \geq 6$, we claim that not only the matrix representing (on the same bases used above) ${}_MT_{|_M\tilde{P}_a|^2}\big|_{\mathcal{F}_4}$ is lower triangular, but also that the only non-zero entry on the diagonal is the element $(1,1)$. This means we have to prove that

$$ {}_MT_{|_M\tilde{P}_a|^2}\big|_{\mathcal{F}_4}(\cos^l(\cdot)) \tag{6.17}$$

is a (cosine) polynomial of degree strictly less than l, for each $l \in \{1,\dots,4\}$. To show this, it is enough to observe that $|_M\tilde{P}_a|^2$ has degree 4, hence (6.17) is a cosine polynomial of degree less than or equal to $\lfloor\frac{4+l}{M}\rfloor < l$. Hence the only non-zero eigenvalue is equal to the entry $(1,1)$ in the matrix, which is (α_M is as in (3.7))

$$\begin{aligned}\rho\big({}_MT_{|_M\tilde{P}_a|^2}\big|_{\mathcal{F}_4}\big) =&\frac{M}{8}\Big(\frac{35\alpha_M^2+7a^4+19a^2-14a^3\alpha_M-46a\alpha_M+7a^2\alpha_M^2}{a^2}+\\ &+\frac{-30a\alpha_M^2-22a^3+52\alpha_Ma^2}{a^2}\Big)\end{aligned}$$

In particular one can find the interval of admissibility for every M by determining α_M and solving an equation of order 4.

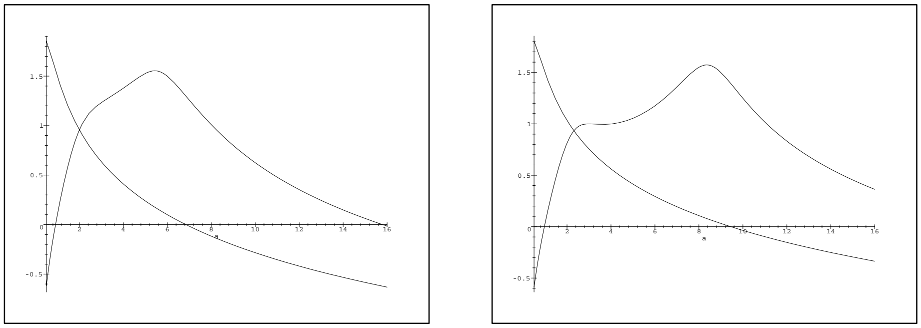

FIGURE 17. Sobolev exponent of primal (monotone decreasing) and dual scaling functions, $M = 4, 5$

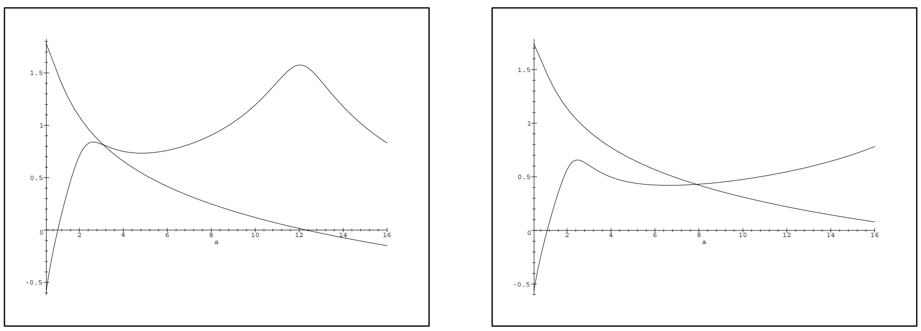

FIGURE 18. Sobolev exponent of primal (monotone decreasing) and dual scaling functions, $M = 6, 8$

We plot in Fig.17-18 the graph representing the Sobolev regularity of the primal and dual scaling functions for $M \in \{4, 5, 6, 8\}$.

7. Comments and Applications

Our construction takes advantage of both the flexibility of the biorthogonal setting and the M-band structure in order to design simple M-band filters and different families of wavelets which enjoy both good localization and symmetry properties.

Some rather recent works are related to ours. Chui and Lian present in [6] some families of 3-band symmetric/antisymmetric orthonormal wavelets with good (and possibly arbitrarily high) regularity. Bi, Dai and Sun presented in [3] M-band orthonormal, compactly supported, cardinal scaling function, for $M \geq 3$ (no such scaling functions exist for $M = 2$). In [2], Belogay and Wang study families of symmetric orthonormal scaling functions for any dilation M: some filters are given in explicit form, and the regularity of the associated wavelets is determined. Here, in the biorthogonal setting, we were able to carry out a completely explicit construction for any M, with very simple filters. Our scaling functions have poor flexibility as far as smoothness is regarded, but one can take advantage of the freedom in choosing a parameter corresponding to wavelets better suited to a particular application.

We have in mind various applications in which these new families of wavelets, in particular $3, 4$-band, are expected to be especially effective. The regularity, vanishing moments, and support lengths of our wavelets seem to be particularly suited for digital sound and image processing, but the combination of all these rather common properties with the symmetry/antisymmetry of our wavelets should prove very important. In fact, these properties imply that an analysis performed by our wavelets will efficiently split symmetric features of a signal from antisymmetric ones. For example, since the human visual perception system seems to be less sensitive to symmetric errors than to antisymmetric ones, improved (lossy) compression could be possibly achieved by discarding "symmetric" and "antisymmetric" coefficients in a selective manner. Moreover, one expects that near an edge, the coefficients of the antisymmetric wavelets and those of the symmetric ones will behave quite differently, which could be used to effectively find edges in images.

8. Acknowledgments

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I dedicate this work to Elena.

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