WAVELET FRAMES ON GROUPS AND HYPERGROUPS VIA DISCRETIZATION OF CALDERÓN FORMULAS

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Abstract. Continuous wavelets are often studied in the general framework of representation theory of square-integrable representations, or by using convolution relations and Fourier transforms. We consider the well-known problem whether these continuous wavelets can be discretized to yield wavelet frames. In this paper we us Calderón-Zygmund singular integral operators and atomic decompositions on spaces of homogeneous type, endowed with families of general translations and dilations, to attack this problem, and obtain strong convergence results for wavelets expansions in a variety of classical functional spaces and smooth molecule spaces. This approach is powerful enough to yield, in a uniform way, for example, frames of smooth wavelets for matrix dilations in $\mathbb{R}^n$, for an affine extension of the Heisenberg group, and on many commutative hypergroups.

1. Introduction

The study of the interplay between continuous and discrete representations of functions is of course classical in analysis. In wavelet theory, this problem becomes the problem of establishing the relationships between continuous and discrete wavelet representations. In this paper we assume we are given a continuous wavelet on some rather general space and seek a discretization of such a continuous family yielding a discrete frame.

This problem has been widely studied in the literature. Early unifying results in this direction, for wavelet and Gabor systems, can be found in [12]. Very general results, in connection with irregular sampling problems, have been proved in a series of papers by Feichtinger and Gröchenig [16, 17, 18] and, recently, with a group-theoretic approach, in the context of representations of Lie groups, in [3, 4] and references therein contained. Constructions of orthonormal, smooth and localized wavelets on (stratified) Lie groups have been carried out in [37].

In this paper we approach the discretization problem with techniques different from the above. It is very well-known that the operators obtained by partial decomposition on (suitable) wavelet bases are associated with Calderón-Zygmund singular integral operators . This observation and the well-established theory of these integral operators allowed one to obtain general convergence results for discrete wavelet expansions (in one dimension). At this stage it already became clear the role of wavelets as "universal atoms" [9]. Higher dimensions and general dilations in multi-dimensional spaces offered new problems, many of which are still open. More recently it has been shown how the theory of Calderón-Zygmund singular integral operators can be applied to the discretization of continuous, classical wavelets in one dimension [30] to yield strong convergence results in smoothness spaces. In this paper we generalize the approach taken in [30], to show how the theory of Calderón-Zygmund singular integral operators on spaces of homogeneous type can be used to obtain results of discretization of continuous wavelets and general strong convergence results for wavelets in higher dimensions and on many groups, among which, notably, the Heisenberg group, and hypergroups. In fact, our technique does not rely on any algebraic structure on the underlying space, but only on the existence of suitable families of translation and dilations operators (not necessarily arising from geometric or algebraic actions), and of an associated continuous wavelet transform (whose origin may be algebraically motivated or not, but we are not concerned with this here). Our convergence results are in most classical functional spaces (as defined on spaces of homogeneous type) whose definition does not depend on the representation (as in the case of the co-orbit spaces of Feichtinger and Gröchenig [17]) and also in smoothness spaces.

Since continuous wavelets, with desirable properties such as smoothness and time/frequency localization, can be constructed in quite general situations ([6]),[15], [20],[22],[23],[24],[32],[34],[33],[35],[42],[43]) our results allow to deduce the existence of wavelet frames with similar properties. The convergence of these wavelet expansions occurs in most classical function spaces, Besov spaces, and smoothness spaces, and are thus an ideal tool for analyzing families of integral operators that are almost diagonal in these bases.

In Section 2 we quickly review some aspects of the theory of Calderón-Zygmund integral operators on spaces of homogeneous type, in Section 3 we introduce the notion of "admissible" translations and dilations operators on such spaces, from which the notion of wavelets arises. At that point we will have all the notions we need to state the main result and sketch the steps of its proof. In Section 4 we cite some of the many settings (e.g. $\mathbb{R}^n$, the Heisenberg group, the Bessel hypergroup) to which our results can be uniformly applied. The main ingredients and steps of the proof of the main theorem take the following Sections 5, 6, 7, and the technical details are left to the two Appendices.

2. Calderón-Zygmund Singular Integral Operators and Molecular Spaces on Spaces of Homogeneous Type

In this section we would like to set the notation and recall many results about Calderón-Zygmund operators on spaces of homogeneous type that we will use in all that follows.

Let G be a set; $\rho : G \times G \to [0, +\infty)$ is said to be a quasi-distance on G if

(i) $\rho(x, y) = 0$ if and only if $x = y$, for any x, y in G.
(ii) $\rho(x, y) = \rho(y, x)$ for any x, y in G.
(iii) There exists C_ρ such that for any x, y, z in G we have

$$\rho(x, y) \leq C_\rho(\rho(x, z) + \rho(z, y)). \tag{2.1}$$

If ρ is a quasi-distance on G, we say that (G, ρ) is a quasi-metric space.

A triple (G, ρ, μ) is a space of homogeneous type ([10]) if (G, ρ) is a quasi-metric space and μ is a positive, non-atomic measure on the σ-algebra generated by the ρ-open sets and if there exists a constant $C > 0$ such that for all x in G and all positive r we have

$$0 < \mu\left(B(x, 2r)\right) \leq C\mu\left(B(x, r)\right) < \infty.$$

Spaces of homogeneous type are natural ambient spaces for Calderón-Zygmund operators. It is proved in [39] (see also [14]) that the "volume function"

$$V(x, y) = \inf\{r > 0 : y \in B(x, r)\}$$

satisfies $V(x, y) = C\rho(x, y)^{\mathrm{n}}$ for some $\mathrm{n} > 0$ (of course, not necessarily integer), and that we can replace the quasi-metric ρ by a topologically equivalent quasi-metric $\tilde{\rho}$ satisfying the following Lipschitz condition: there exists C and $\theta \in (0, \mathrm{n})$ such that for all $x, y, z \in G$

$$|\tilde{\rho}(x, z) - \tilde{\rho}(y, z)| \leq C\tilde{\rho}(x, y)^\theta \left(\tilde{\rho}(x, z) + \tilde{\rho}(y, z)\right)^{\mathrm{n}-\theta}.$$

From now on we will use this new metric, and in the following we shall write ρ instead of $\tilde{\rho}$.

Lipschitz spaces are natural test spaces. A complex function f (modulo constants) on G is said to belong to the Lipschitz space $\mathcal{C}^\eta(G)$, $\eta < \theta$, if

$$||f||_\eta := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\eta} < \infty \ .$$

We denote by $\mathcal{C}_0^\eta(G)$ the space of η-Lipschitz functions which are compactly supported. The smoothness of the metric ρ guarantees that $\mathcal{C}_0^\eta(G)$ is dense in $\mathbb{L}^2(G)$ if η is small enough [39].

We now introduce Calderón-Zygmund integral operators.

Definition 2.1. A kernel $K : (G \times G) \backslash \{(x,x) : x \in G\} \to \mathbb{C}$ is a Calderón-Zygmund kernel if there exist $\epsilon > 0$ and $C > 0$ such that

$$|K(x,y)| \leq \frac{C}{\rho(x,y)^{\mathrm{n}}}$$

$$|K(x,y) - K(x',y)| \leq C\rho(x,x')^\epsilon \rho(x,y)^{-(\mathrm{n}+\epsilon)}$$

for $\rho(x,x') \leq \frac{1}{2}\rho(x,y)$, and

$$|K(x,y) - K(x,y')| \leq C\rho(y,y')^\epsilon \rho(x,y)^{-(\mathrm{n}+\epsilon)}$$

for $\rho(y,y') \leq \frac{1}{2}\rho(x,y)$.

Definition 2.2. A continuous linear operator $T : \mathcal{C}_0^\eta(G) \to (\mathcal{C}_0^\eta(G))'$ is said to be a Calderón-Zygmund singular integral operator if it is associated to a Calderón-Zygmund kernel K in the sense that

$$< T\phi, \psi >= \int_G \int_G K(x,y)\phi(x)\psi(y)dxdy$$

for all $\phi, \psi \in \mathcal{C}_0^\eta(G)$ with $\mathrm{supp}.\phi \cap \mathrm{supp}.\psi = \emptyset$. The smallest constant C for which the estimates on K in Definition 2.1 hold is called the Calderón-Zygmund norm of T, and will be denoted by $||T||_{CZ}$.

With techniques similar to the ones used in the Euclidean case, one proves that the $\mathbb{L}^2$-boundedness of a Calderón-Zygmund singular integral operator implies the boundedness on all $\mathbb{L}^p$ spaces:

Theorem 2.1. If T is a Calderón-Zygmund singular integral operator which is bounded on $\mathbb{L}^2(G)$, then T is also bounded on $\mathbb{L}^p(G)$ for $1 < p < \infty$, is of weak type $(1,1)$ and maps $\mathbb{L}^\infty(G)$ boundedly into $\mathrm{BMO}(G)$.

The problem of the boundedness of a Calderón-Zygmund singular integral operator is thus reduced to the problem of $\mathbb{L}^2$-boundedness.

Definition 2.3. A Calderón-Zygmund singular integral operator is said to be weakly bounded if there exists $\eta > 0$ and C so that

$$|< T\phi, \psi >| \leq Cr^{\mathrm{n}}$$

for all $\phi, \psi \in \mathcal{C}_0^\eta(G)$ supported in $B(x,r)$ with $||\phi||_\infty, ||\psi||_\infty \leq 1$ and $||\phi||_\eta, ||\psi||_\eta \leq r^{-\eta}$.

The decay and cancellation properties of Calerón-Zygmund kernels allow one to define $T1$ and T^*1 in the sense of distributions. We then have a version of the celebrated David-Journé's "T1 Theorem" [13] for spaces of homogeneous type:

Theorem 2.2. A Calderón-Zygmund singular integral operator T has an extension as a continuous linear operator on $\mathbb{L}^2(G)$ if and only if $T1 \in \mathrm{BMO}$, $T^*1 \in \mathrm{BMO}$ and T is weakly bounded.

We would like to have results about the boundedness of (special) Calderón-Zygmund singular integral operators on spaces of smooth functions, which will guarantee the smoothness of dual functions. To begin with, we define molecular spaces.

Definition 2.4. A function f on the space of homogeneous type (X, ρ, μ), with $V(x,y) \cong \rho(x,y)^{\mathrm{n}}$, is said to belong to $\mathcal{M}(x,r,\beta,\gamma)$, the space of molecules of center x, radius r and parameters β, γ, if there exists a constant C such that the following inequalities are satisfied:

(i) Decay condition: for all $y \in X$

$$|f(y)| \le C \frac{r^\gamma}{(r+\rho(y,x))^{\mathrm{n}+\gamma}} \ . \tag{2.2}$$

(ii) Smoothness condition: for all $y_1, y_2 \in X$ with $\rho(y_1,y_2) \le r + \rho(x,y_1)$

$$|f(y_1) - f(y_2)| \le C \frac{\rho(y_1,y_2)^\beta r^\gamma}{(r+\rho(y_1,x))^{\mathrm{n}+\gamma}} \ . \tag{2.3}$$

(iii) Vanishing moment condition:

$$\int_X f(y) d\mu(y) = 0 \ . \tag{2.4}$$

The smallest constant C such that (i) and (ii) hold is denoted by $||f||_{\mathcal{M}(x,r,\beta,\gamma)}$ and is a norm with respect to which $\mathcal{M}(x,r,\beta,\gamma)$ is a Banach space.

We observe that

(a) Condition (i) guarantees that $\mathcal{M}(x,r,\beta,\gamma) \subset \mathbb{L}^1(X,\mu)$. Moreover

$$||f||_{\mathbb{L}^1(X,\mu)} \le 2$$

for any molecule f.

(b) Condition (ii) is a Lipschitz condition which grows stronger with the distance from the center of the molecule.

(c) It is clear that $\mathcal{M}(x_0,r_0,\beta,\gamma) = \mathcal{M}(x_1,r_1,\beta,\gamma)$ with equivalent norms.

Before stating an analogue of the "T1 Theorem" for these molecular spaces, we need the following definition:

Definition 2.5. [26] An operator $T : \mathcal{C}_0^\eta(G) \to (\mathcal{C}_0^\eta(G))'$, with kernel K, has the strong weak boundedness property if there exists $\eta > 0$ and C such that

$$\left| < K, f >_{\mathbb{L}^2(G\times G)} \right| \le C r^{\mathrm{n}} \tag{2.5}$$

for all $f \in \mathcal{C}_0^\eta(G \times G)$ with supp.$f \subset B(x_1,r) \times B(y_1,r)$, $x_1, y_1 \in G$, $||f||_\infty \le 1$, $||f(\cdot,y)||_\eta \le r^{-\eta}$, $||f(x,\cdot)||_\eta \le r^{-\eta}$ for all $x, y \in G$.

Theorem 2.3. [26] Suppose T is a Calderón-Zygmund singular integral operator such that $T(1) = T^*(1) = 0$ (in BMO) and T is strongly weakly bounded. Furthermore, suppose that $K(x,y)$, the kernel of T, satisfies the following smoothness condition

$$|[K(x,y) - K(x',y)] - [K(x,y') - K(x',y')]| \le C\rho(x,x')^\epsilon \rho(y,y')^\epsilon \rho(x,y)^{-(1+2\epsilon)} \tag{2.6}$$

for $\rho(x,x') \le \frac{1}{4}\rho(x,y)$ and $\rho(y,y') \le \frac{1}{4}\rho(x,y)$. Then T maps $\mathcal{M}(\beta,\gamma)$ into itself for $0 < \beta, \gamma < \epsilon$, and there exists a constant $C > 0$ (not depending on T) such that

$$||Tf||_{\mathcal{M}(\beta,\gamma)} \le C ||T||_{CZ} ||f||_{\mathcal{M}(\beta,\gamma)}.$$

3. Translations, Dilations, and Wavelets

Since we will focus completely on molecules as building blocks, we will define translation and dilation operators only with respect to their action on molecules, disregarding any geometric interpretation. This approach will allow us to embrace very different situations and consider wavelets in $\mathbb{R}^n$, on many groups and hypergroups, all at once.

Let (X, ρ, μ) be a space of homogeneous type, with an involution $\cdot \mapsto \cdot^-$, i.e. an homeomorphism onto which satisfies $(x^-)^- = x$ for every x in X.

3.1. **Translations.** A family of admissible translations is a set of operators

$$\{T_x\}_{x \in X}$$

on $\mathcal{C}(X)$ such that:

(T1) For every x in X, T_x maps $\mathcal{M}(0, r, \beta, \gamma)$ into $\mathcal{M}(x^-, r, \beta, \gamma)$, and

$$||T_x||_{\mathcal{M}(0,r,\beta,\gamma) \to \mathcal{M}(x^-,r,\beta,\gamma)} \le C_T,$$

for some C_T depending only on β and γ;

(T2) For every x, y in X we have

$$T_x f(y) = T_y f(x).$$

(T3) For every x in X and $f \in \mathcal{M}(0, r, \beta, \gamma)$ we have $(T_x f)^- = T_{x^-} f^-$, where $f^-(\cdot) = f(\cdot^-)$.

Remark 3.1. We will suppose $C_T \le 1$: from the proofs it will be clear that we can do so without loss of generality, since it only scales all our estimates by a constant factor.

3.2. **Dilations.** A family of (generalized) dilations is a set of operators

$$\{D_\delta\}_{\delta > 0}$$

such that:

(D1) For all $\delta > 0$, D_δ maps $\mathcal{M}(0, r, \beta, \gamma)$ into $\mathcal{M}(0, \delta r, \beta, \gamma)$;

(D2) For all $\delta > 0$ we have $(D_\delta(f))^- = D_\delta(f^-)$.

(D3) There exists constants $C_{D,\beta,\gamma}, \mathrm{d} > 0$ such that for all $\delta, \eta > 0$ with $\left|\frac{\delta}{\eta} - 1\right| < 2$, we have

$$\|D_\delta - D_\eta\|_{\mathcal{M}(0,r,\beta,\gamma) \to \mathcal{M}(0,\delta r,\beta,\gamma)} \le C_D \left|1 - \frac{\delta}{\eta}\right|^{\mathrm{d}},$$

with C_D depending only on β and γ.

3.3. **Wavelets.** From now on we will work with a space of homogeneous type (X, ρ, μ) with an involution $\cdot^-$, together with a family of admissible translations $\{T_x\}_{x \in X}$ and dilations $\{D_\delta\}_{\delta > 0}$.

We can define the continuous wavelet family generated by a function ψ in two ways:

(W1) If $\rho(x, y) = \rho(x^-, y^-)$ for all x, y in X and ψ is in $\mathcal{M}(0, r, \beta, \gamma)$, we can define

$$\psi_{\delta,x} = T_x D_\delta \psi.$$

(W2) For ψ in $\mathcal{M}(0, r, \beta, \gamma)$ such that $\tilde{\psi}(\cdot) := \psi(\cdot^-)$ is also in $\mathcal{M}(0, r, \beta, \gamma)$, we can define

$$\psi_{\delta,x} = T_{x^-} D_\delta \psi.$$

In the context of hypergroups, definition (W1) seems more customary, in most other situations (W2) is used. The difference between the two definitions arise only in non-commutativity or non-unimodular settings.

Definition 3.1. If the reproducing formula

$$f = \int_0^{+\infty} \int_X < f, \psi_{\delta,y} > \psi_{\delta,y} d\mu(y) \frac{d\delta}{\delta}, \tag{3.1}$$

holds for f in $\mathbb{L}^2(X,\mu)$ (or an appropriate subspace), then ψ is called *admissible.*

We will see momentarily that in many different situations (for many groups and hypergroups, in coherent state representations, etc...) there exist a large set of *admissible* ψ for which of one is able to prove continuous reproducing formulas in the form (3.1). It is not a serious restriction in general to require that an admissible ψ is in $\mathcal{M}(0,r,\beta,\gamma)$ for some $r,\beta,\gamma > 0$, and, if wavelets are defined as in (W2), that $\tilde{\psi}$ also belongs to $\mathcal{M}(0,r,\beta,\gamma)$, so we will assume this in what follows.

The goal of discretization is to sample $X \times (0,+\infty)$ at points $\{\{(x_{j,k},\delta_j)\}_{k\in\mathcal{K}(j)}\}_{j\in\mathbb{Z}}$ and find constants $\lambda_{j,k}$ such that the system

$$\left\{\{\lambda_{j,k}\psi_{\delta_j,x_{j,k}}\}_{k\in\mathcal{K}(j)}\right\}_{j\in\mathbb{Z}}$$

is a frame for the subspace generated by $\{\psi_{\delta,x}\}_{x\in X,\delta>0}$. This problem is related to function sampling and their discrete representation, atomic decompositions, irregular sampling problems, discretization of coherent states, frame construction, existence of discrete wavelets.

We proceed as follows. We discretize the dilations by choosing $\delta_j = a^j$ for some fixed $a > 0$. For the translations, fixed a positive b, at each scale j we choose a countable covering $\{Q^{(a,b)}_{j,k}\}_{k\in\mathcal{K}(j)}$ of X such that

(i) There exist two constants $c_1, c_2 > 0$ and a set of points $\{z_{j,k}\}_{k\in\mathcal{K}(j)}$ such that, at every scale,

$$B(z_{j,k}, c_1 a^{-j} b) \subset Q^{(a,b)}_{j,k} \subset B(z^{(a,b)}_{j,k}, c_2 a^{-j} b).$$

The points $z^{(a,b)}_{j,k}$'s will be fixed with $\{Q^{(a,b)}_{j,k}\}$ and called centers of the $Q^{(a,b)}_{j,k}$'s.

(ii) Almost every point in X does not belong to more than N_j of the $\{Q^{(a,b)}_{j,k}\}_{k\in\mathcal{K}(j)}$'s. When we consider families of coverings, depending on (a,b), we shall assume that N_j can be chosen independent of a,b for all a close to 1 and b close to 0 from above.

Remark 3.2. It will be clear from the proofs that we can assume without loss of generality $N_j = N = 1$. The purpose of $N_j > 1$ above is to allow partial overlapping of the cells $Q^{(a,b)}_{j,k}$ at each resolution j. This implies our results apply to the "quasi-affine" setting, and we think it is also of independent interest.

The candidate wavelets are then

$$\psi^{(a,b)}_{j,k}(\cdot) = \psi_{\delta_j, z^{(a,b)}_{j,k}} \tag{3.2}$$

with weights

$$\lambda^{(a,b)}_{j,k} = \left(\frac{1}{N}\mu\left(Q^{(a,b)}_{j,k}\right)\right)^{\frac{1}{2}} \tag{3.3}$$

Then we have the following:

Theorem 3.1. If ψ is admissible and in $\mathcal{M}(0,r,\beta,\gamma)$, with the assumptions and definitions of (W1) or (W2), then for every $\beta' < \beta$, there exist $a > 1$ and $b > 0$ such that $\{\lambda^{(a,b)}_{j,k}\psi^{(a,b)}_{j,k}\}$ is a frame for the closure of subspace of $\mathbb{L}^2(X,\mu)$ generated by $\{\psi_{\delta,b}\}_{\delta>0,b\in X}$. Moreover, if $\mathcal{B}$ is $\mathcal{M}(0,r,\beta'',\gamma'')$ for some $\beta'' < \beta', \gamma'' < \gamma$, or

$\mathbb{L}^p(X,\mu)$ for some $1 < p < +\infty$, there exists a system of dual functions $\phi_{j,k}^{(a,b)}$ in $\mathcal{M}(0,r,\beta',\gamma)$ such that we have

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} < f, \psi_{j,k}^{(a,b)} > \phi_{j,k}^{(a,b)} \tag{3.4}$$

$$= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} < f, \phi_{j,k}^{(a,b)} > \psi_{j,k}^{(a,b)} \tag{3.5}$$

for every f in the closure of the subspace of $\mathcal{B}$ generated by $\{\psi_{\delta,b}\}_{\delta>0,b\in X}$, with convergence in $\mathcal{B}$.

Remark 3.3. At least when $D_{a^j}^{-1} = D_{a^{-j}}$, we have $\phi_{j,k}^{(a,b)} = D_{a^{-j}}\phi_{0,k}^{(a,b)}$. In general $\phi_{j,k}^{(a,b)}$ is not a translate of $\phi_{j,0}^{(a,b)}$. It is so when $S^{(a,b)}$ commutes with $T_{z_{j,k}}^{(a,b)}$ and the involution is a bijection of $\{z_{j,k}^{(a,b)}\}_{k\in\mathcal{K}(j)}$ for every j.

The proof of Theorem 3.1 consists of the following steps:

(1) First we "guess" a discrete reproducing formula with the data at hand: we try with the "Riemann sum" operator (recall we are assuming, without loss of generality, $N = 1$ in (ii) above)

$$S^{(a,b)} = \ln a \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} < f, \mu(Q_{j,k}^{(a,b)})^{\frac{1}{2}} \psi_{j,k}^{(a,b)} > \mu(Q_{j,k}^{(a,b)})^{\frac{1}{2}} \psi_{j,k}^{(a,b)}(x). \tag{3.6}$$

By using quasi-orthogonality properties of molecules across translations and dilations, and Littlewood-Paley theory, we prove $S^{(a,b)}$ is well defined and bounded on $\mathbb{L}^2(X,\mu)$.

(2) Secondly, we want to show that $S^{(a,b)}$ is invertible, in $\mathbb{L}^2(X,\mu)$ but also in many molecular spaces, for all $a > 1$ sufficiently close to 1 and $b > 0$ sufficiently close to 0. In order to do so, we look at the difference between $S^{(a,b)}$ and the identity operator decomposed according to the continuous reproducing formula (3.1). We show that this difference can be decomposed in a sum of nice Calderón-Zygmund singular integral operators , whose norm goes to 0 as $a \to 1^+$ and $b \to 0^+$.

(3) For these values of a, b for which $S^{(a,b)}$ is invertible, we can now define $\phi_{j,k}^{(a,b)} = \mu(Q_{j,k}^{(a,b)})^{\frac{1}{2}} \left(S^{(a,b)}\right)^{-1} \psi_{j,k}^{(a,b)}$, and we are left with showing that the two systems are dual frames and that the discrete expansions converge in the relevant functional spaces.

Before presenting the details of the proof, we would like to present a carousel of examples to which Theorem 3.1 can be applied.

4. Examples and Applications

4.1. **Hypergroups.** In this section we would like to remind the basic definitions in the theory of hypergroups, and how one introduces a generalized continuous wavelet transform on them. Our main reference is [41], and we direct the interested reader there, for details and a rich list of further references.

Definition 4.1. Let X be a nontrivial locally compact Hausdorff space, and $*_X$ a binary operation which turns $M^b(X)$, the space of bounded Radon measures on X, into an algebra. The pair $(X, *_X)$ is an *hypergroup* if

(H1) The mapping

$$\begin{array}{rcl} X \times X & \to & M^1(X) \\ (x,y) & \mapsto & \delta_x *_X \delta_y \end{array}$$

is continuous (here $M^1(X)$ denotes subset of $M^b(X)$ consisting of all probability measures).

(H2) There exists an element e in X such that
$$\delta_e *_X \delta_x = \delta_x *_X \delta_e = \delta_x$$
for every x in X.

(H3) There exists an involutive homeomorphis $x \mapsto x^-$ of X onto itself, with the property $(x^-)^- = x$ for all x in X, such that
$$(\delta_x *_X \delta_y)^- = \delta_{y^-} *_X \delta_{x^-},$$
where $\mu^-(E) = \mu(E^-)$.

(H4) For all x, y in X, e is in the support of $\delta_x *_X \delta_y$ if and only if $x = y^-$.

(H5) The support of $\delta_x *_X \delta_y$ is compact for every x, y in X.

(H6) The mapping
$$\begin{array}{rcl} X \times X & \rightarrow & \mathcal{K}(X) \\ (x, y) & \mapsto & \text{supp.}(\delta_x *_X \delta_y) \end{array}$$

is continuous, where $\mathcal{K}(X)$ is the space of nonvoid compact subsets of X endowed with the Michael topology, which has a sub-basis
$$\{K \in \mathcal{K}(X) : K \cap U \neq \emptyset \text{ and } K \subset V\}_{U,V \text{ open in } X}.$$

Remark 4.1. All the hypergroups we will consider will be endowed with a metric ρ, in which case the Michael topology is equivalent to the topology induced by the Hausdorff metric on $\mathcal{K}(X)$, defined, for A and B in $\mathcal{K}(X)$, by
$$d_H(A, B) = \inf \{r : A \subset V_r(B) \text{ and } B \subset V_r(A)\},$$
where $V_r(E)$ is a r-neighborhood of E, i.e. $V_r(E) = \{y \in X : \exists x \in E : \rho(x, y) < r\}$.

Definition 4.2. A positive Radon measure μ on an hypergroup $(X, *_X)$ is a right (resp. left) Haar measure for that hypergroup if
$$\int_X \left(\delta_x *_X \delta_{y^-}\right)(f) d\mu(y) = \int_X f(y) d\mu(y)$$
$$(\text{resp.} \int_X \left(\delta_{y^-} *_X \delta_x\right)(f) d\mu(y) = \int_X f(y) d\mu(y)\ )$$

There exists a right Haar measure on every compact or commutative hypergroup, and such a measure is unique up to a positive multiplicative constant, its support is X and is bounded only if the hypergroup is compact.

A hypergroup $(X, *_X)$ is said to be *commutative* if $(M^b(X), *_X)$ is a commutative algebra. In the rest of this section, let $(X, *_X)$ be a commutative hypergroup with right Haar measure μ_X.

Definition 4.3. We define the *generalized translation operators* T_x, x in X, on $\mathcal{C}(X)$ by
$$T_x f(y) = \int_X f(z) \left(\delta_x *_X \delta_y\right)(z) \tag{4.1}$$

The generalized translation operators have the following properties:

(i) For all x, y, z in X and all f in $\mathcal{C}(X)$
$$T_e f(x) = f(x)$$
$$T_x f(y) = T_y f(x)$$
$$T_x T_y f(z) = T_y T_x f(z)$$

(ii) The mapping $(x, y) \mapsto T_x f(y)$ is continuous on $X \times X$ for all $f \in \mathcal{C}_b(X)$.
(iii) If f is in $\mathcal{C}_c(X)$, then $T.f(x)$ is in $\mathcal{C}_c(X)$ for all x in X.

Definition 4.4. For μ, ν in $M^b(X)$ we define the *generalized convolution* of μ and ν by

$$\mu *_X \nu(f) = \int_X \int_X T_x f(y) d\mu(x) d\nu(y) \tag{4.2}$$

for every f in $\mathcal{C}_c(X)$.

If f, g are in $\mathbb{L}^1(X, \mu_X)$ and $\mu = f\mu_X$ and $\nu = g\mu_X$, then

$$\mu *_X \nu = (f *_X g)\mu_X,$$

where

$$(f *_X g)(x) = \int_X T_x f(y) g(y) d\mu_X(y) = \int_X f(y) T_{x^-} g(y) d\mu_X(y) \tag{4.3}$$

is the generalized convolution product of f and g.

Definition 4.5. A *character* χ is a nonzero homomorphism of the algebra $(\mathbb{L}^1(X, \mu_X), *_X)$ into $\mathbb{C}$, i.e. $\chi \in \mathcal{C}_b(X)$ and $\forall x, y, \in X$

$$T_x \chi(y) = \chi(x)\chi(y).$$

A character is hermitian if $\chi(x^-) = \overline{\chi(x)}$ for all x in X.

The *dual* $\hat{X}$ of X is the weakly closed part of the Gelfand maximal ideal of $\mathbb{L}^1(X, \mu_X)$ consisting of the hermitian characters of X. In general $\hat{X}$ can not be endowed with a convolution $*_{\hat{X}}$ which makes it a hypergroup. If it does, then $X \subset \hat{\hat{X}}$, and if $X = \hat{\hat{X}}$ then X is called a strong hypergroup.

The generalized Fourier transform of a measure μ in $M^b(X)$ is defined by

$$\mathcal{F}_X(\mu)(\chi) = \int_X \overline{\chi(x)} d\mu(x) \tag{4.4}$$

for χ in $\hat{X}$.

The Fourier transform has the following properties:

(i) The function $\mathcal{F}_X(\mu)$ is continuous on $\hat{X}$ for every μ in $M^b(X)$, and $||\mathcal{F}_X(\mu)||_\infty \leq ||\mu||_{M^b(X)}$.
(ii) For every μ, ν in $M^b(X)$

$$\mathcal{F}_X(\mu *_X \nu) = \mathcal{F}_X(\mu)\mathcal{F}_X(\nu).$$

(iii) If f is in $\mathbb{L}^1(X, \mu_X)$, then $\mathcal{F}_X(f)$ is continuous on $\hat{X}$, tends to zero at infinity and $||\mathcal{F}_X(f)||_\infty \leq ||f||_1$.
(iv) There exists a unique positive Radon measure π on $\hat{X}$, called the Plancherel measure of X, such that

$$< f, g >_{\mathbb{L}^2(X, \mu_X)} = < f, g >_{\mathbb{L}^2(\hat{X}, \pi)},$$

for every f, g in $\mathbb{L}^2(X, \mu_X)$.

4.1.1. *Examples.*

(i) $(G, \cdot_G, \mu_G)$, where G is a locally compact abelian group with operation $\cdot_G$ and Haar measure μ_G, and involution defined by $g \mapsto g^{-1}$ $(g \in G)$, is a hypergroup when the translation operators are defined by

$$T_x f(y) = f(x \cdot_G y)$$

and the convolution is as prescribed by (4.2).

(ii) $([0,+\infty),*_X,dx)$, with involution given by the identity mapping, where the convolution $*_X$ is defined as in (4.2), with generalized translations

$$T_x f(y) = \frac{1}{2}\left(f(x+y) + f(|x-y|)\right),$$

is a hypergroup. The characters are

$$\chi_\lambda(x) = \cos(\lambda x),$$

with λ in $[0,+\infty)$. The natural dilations are

$$\left(D_\delta f\right)(x) = \frac{1}{\delta} f\left(\frac{x}{\delta}\right).$$

(iii) $([0,+\infty),*_\alpha,x^{2\alpha+1}dx)$, for some $\alpha > -\frac{1}{2}$. With involution defined by the identity mapping, and translation operators defined by

$$\begin{aligned} T_x f(y) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos\theta}\right)(\sin\theta)^{2\alpha} d\theta \\ &= \int_0^{+\infty} f(z) W(x,y,z) d\mu_\alpha(z) \end{aligned}$$

where

$$W(x,y,z) = \frac{2^{1-2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \frac{\left[(x+y)^2-z^2\right]^{\alpha-\frac{1}{2}}\left[z^2-(x-y)^2\right]^{\alpha-\frac{1}{2}}}{(xyz)^{2\alpha}} \chi_{(|x-y|,x+y)}(z).$$

Let us denote with $*_\alpha$ the convolution associated to these translations as in (4.2). Then $((0,+\infty),*_\alpha,x^{2\alpha+1}dx)$ is a hypergroup, called the Bessel-Kingman hypergroup of order α. The characters are

$$\chi_\lambda(x) = j_\alpha(\lambda x),$$

for λ in $[0,+\infty)$ and

$$j_\alpha(\lambda x) = \begin{cases} 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(\lambda x)}{(\lambda x)^\alpha} & ,\lambda x \neq 0 \\ 1 & ,\lambda x = 0 \end{cases},$$

J_α being the Bessel function of first kind and index α. These translations are admissible:

Lemma 4.1. For any $\alpha > -\frac{1}{2}$, the family $\{T_x\}_{x>0}$ of translation operators of the Bessel-Kingman hypergroup of order α satisfies the conditions (i) and (ii) above.

Proof. Fix $f \in \mathcal{M}(0,r,\beta,\gamma)$. For the decay condition we have the estimate:

$$\begin{aligned} |T_x f(y)| &\leq \sup_{\theta\in[0,\pi]} \left| f\left(\sqrt{x^2+y^2-2xy\cos\theta}\right)\right| \\ &\leq \sup_{\theta\in[0,\pi]} \left|\frac{r^\gamma}{(r+\sqrt{x^2+y^2-2xy\cos\theta})^{\mathrm{n}+\gamma}}\right| \\ &\leq \frac{r^\gamma}{(r+|x-y|)^{\mathrm{n}+\gamma}}. \end{aligned}$$

For the smoothness condition, when $|y_1-y_2| \leq r + |x-y_1|$ we have

$$|T_x f(y_1) - T_x f(y_2)| \leq$$

$$\leq \sup_{\theta\in[0,\pi]} \left| f\left(\sqrt{x^2+y_1^2-2xy_1\cos\theta}\right) - f\left(\sqrt{x^2+y_2^2-2xy_2\cos\theta}\right)\right|$$

$$\leq \frac{\sup_{\theta\in[0,\pi]} \left|\sqrt{x^2+y_1^2-2xy_1\cos\theta} - \sqrt{x^2+y_2^2-2xy_2\cos\theta}\right|^\beta r^\gamma}{(r+|x-y|)^{\mathrm{n}+\gamma}}$$

(because $|y_1-y_2|$ is also less than $r+|x|$, since y_1 is positive)

$$\leq \frac{|y_1-y_2|^\beta r^\gamma}{(r+|x-y|)^{\mathrm{n}+\gamma}}.$$

The last inequality follows from the triangle inequality applied to the triangle with vertices $A=(0,0)$,$B=(x-y_1\cos\theta, y_1\sin\theta)$,$C=(x-y_2\cos\theta, y_2\sin\theta)$. The moment condition is satisfied by $T_x f$ by definition of Haar measure for a hypergroup. □

The family of dilations

$$D_\delta f(x) = \frac{1}{\delta^{2\alpha+2}}\psi\left(\frac{x}{\delta}\right),$$

with $\delta>0$, is admissible.

4.2. **Continuous Wavelets on Hypergroups.** Let X be a commutative hypergroup with the following properties:

(i) $\hat{X}$ can be identified with a subset Γ of $\mathbb{C}^n$ that includes $\mathbb{R}^n$;

(ii) for any x in X the map

$$\begin{array}{ccc} \Gamma & \to & \mathbb{C} \\ \lambda & \mapsto & \chi_\lambda(x) \end{array}$$

is continuous;

(iii) the Plancherel measure $d\pi$ is absolutely continuous with respect to the Lebesgue measure of $\mathbb{C}^n$, with density π and support S;

(iv) for all λ in S and x in X, $\chi_\lambda(x)$ is real;

(v) π is continuous on S and there exists η in $\mathbb{R}$ such that

$$|\pi(\lambda)| \leq C\left(1+||\gamma||_{\mathbb{C}^n}\right)^\eta;$$

(vi) S is a cone, i.e. it contains 0 and if $a>0$ then $aS=S$;

(vii) the function

$$\begin{array}{rccc} k: & (0,+\infty) & \to & \mathbb{R} \\ & \delta & \mapsto & \sup\limits_{\lambda\in S\setminus\{0\}} \dfrac{\pi(\lambda/\delta)}{\pi(\lambda)} \end{array}$$

is continuous.

Definition 4.6. Let $(X, *_X, \mu)$ be a commutative hypergroup satisfying the conditions (i)-(vii) above. A function ψ in $\mathbb{L}^2(X,\mu)$ is an admissible wavelet if

$$0 < \ C_\psi := \int_0^{+\infty} |\mathcal{F}_X(\psi)(\delta\lambda)|^2 \, \frac{d\lambda}{\lambda} < +\infty \tag{4.5}$$

for a.e. δ in S.

Remark 4.2. Smooth wavelets always exist in this setting: for example define, for $t>0$, α_t by

$$\mathcal{F}_X\alpha_t(\lambda) = \exp\left(-t(||\lambda||^2+||\lambda_0||)\right),$$

for all λ in S. Then the function

$$\psi(x) = -\frac{d}{dt}\alpha_t(x) - ||\lambda_0||^2\alpha_t(x)$$

is a wavelet, with

$$C_\psi = \frac{||\lambda_0||^2 \exp(-2t^2)}{8t^2}.$$

If ψ is a wavelet, then the conditions above imply the existence of a function ψ_δ such that

$$(\mathcal{F}_X \psi_\delta)(\lambda) = \mathcal{F}_X(\psi)(\delta\lambda) \tag{4.6}$$

for all $\delta \in (0, +\infty)$ and all $\lambda \in S$. This means we are forcing Euclidean dilations *on the Fourier side* $\hat{X}$, while, on X, ψ_δ is not necessarily obtained by dilation of ψ in the usual sense. In this way we have exactly what we need to get to Calderón continuous reproducing formula, by using Fourier methods, but has the drawback that we might not have an explicit form for ψ_δ.

If ψ is a function on X, the wavelet family generated by ψ is defined by

$$\psi_{\delta,b}(\cdot) = \sqrt{\delta} T_b \psi_\delta(\cdot) \tag{4.7}$$

The wavelet transform is defined by

$$(\mathcal{W}_\psi f)(\delta, b) = < f, \psi_{\delta,b} > = \sqrt{\delta} \left(f *_X \tilde{\psi_\delta} \right)(b)$$

We recall the following result [41]

Theorem 4.2. Let $(X, *_X, \mu)$ be a commutative hypergroup satisfying the conditions (i)-(vii) above and let ψ be a wavelet on X. Then

(i) For $f \in \mathbb{L}^2(X, \mu)$ and $(\delta, b) \in (0, +\infty) \times X$ we have

$$|(\mathcal{W}_\psi f)(\delta, b)| \leq \left(\frac{k(\delta)}{\delta} \right)^{\frac{1}{2}} ||f||_2 ||\psi||_2.$$

(ii) For $f \in \mathbb{L}^q(X, \mu)$, $q \in [1, 2]$, we have, for $r \in [1, +\infty]$ such that $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$

$$\|(\mathcal{W}_\psi f)(\delta, \cdot)\| \leq \left(\frac{k(\delta)}{\delta} \right)^{\frac{1}{2}} ||f||_q ||\psi||_2.$$

(iii) Plancherel formula for $\mathcal{W}_\psi$:

$$||f||_2^2 = C_\psi^{-1} \int_X \int_0^{+\infty} |(\mathcal{W}_\psi f)(\delta, b)|^2 \frac{d\delta}{\delta} d\mu(b) \tag{4.8}$$

(iv) Calderón reproducing formula: for any f in $\mathbb{L}^2(X)$ we have, weakly in $\mathbb{L}^2(X, \mu)$,

$$\begin{aligned} f &= (C_\psi)^{-1} \int_0^{+\infty} \int_0^{+\infty} (\mathcal{W}_\psi f)(\delta, b) \psi_{\delta,b} d\mu(b) \frac{d\delta}{\delta^2} \\ &= (C_\psi)^{-1} \int_0^{+\infty} < (\mathcal{W}_\psi f)(\delta, \cdot), \sqrt{\delta} T_{(\cdot)} \psi_\delta > \frac{d\delta}{\delta^2} \\ &= (C_\psi)^{-1} \int_0^{+\infty} f *_X \tilde{\psi_\delta} *_X \psi_\delta \frac{d\delta}{\delta} \ . \end{aligned} \tag{4.9}$$

If f belongs to $\mathcal{C}(X) \cap \mathbb{L}^1(X, \mu)$ (resp. $\mathcal{C}(X) \cap \mathbb{L}^2(X, \mu)$) and $\mathcal{F}_X(f)$ belongs to $\mathbb{L}^1(S, \pi)$ (resp. $(\mathbb{L}^1 \cap \mathbb{L}^\infty)(S, \pi)$) then

$$f(x) = (C_\psi)^{-1} \int_0^{+\infty} \int_0^{+\infty} (\mathcal{W}_\psi f)(\delta, b) \psi_{\delta,b} d\mu(b) \frac{d\delta}{\delta^2} \tag{4.10}$$

for every $x \in X$, with the inner integral and the outer integral absolutely convergent, but possibly not the double integral.

(v) Suppose the function

$$\begin{array}{ccc}(0,+\infty) & \to & \mathbb{L}^2(X,\mu)\\ \delta & \mapsto & \psi_\delta\end{array}$$

is continuous. Then $\mathcal{W}_\psi$ is an isometry onto the subspace

$$\mathbb{W}_\psi \subset \mathbb{L}^2\left((0,+\infty)\times X, (C_\psi)^{-1}\frac{d\delta}{\delta}\times d\mu(b)\right)$$

of functions F such that

$$F(\delta,b) = (C_\psi)^{-1}\int_X\int_0^{+\infty} F(\delta',b') < \psi_{\delta',b'},\psi_{\delta,b} > \frac{d\delta'}{\delta'}d\mu(b').$$

In particular, $\mathbb{W}_\psi$ is a reproducing kernel Hilbert space.

We would like to briefly sketch the proof of (iii), all the details can be found [41]. By Fubini-Tonelli we have

$$\begin{aligned} C_\psi^{-1}\int_X\int_0^{+\infty} &\left|\left(\mathcal{W}_\psi f\right)(\delta,b)\right|^2\frac{d\delta}{\delta}d\mu(b) = \\ &= C_\psi^{-1}\int_0^{+\infty}\left(\int_X\left|f *_X \overline{\psi_\delta}\right|^2(b)d\mu(b)\right)\frac{d\delta}{\delta}\\ &= C_\psi^{-1}\int_0^{+\infty}\left(\int_S\left|\mathcal{F}_X(f)(\lambda)\right|^2\left|\mathcal{F}_X(\overline{\psi_\delta})(\lambda)\right|^2\pi(\lambda)d\lambda\right)\frac{d\delta}{\delta}\end{aligned}$$

(by the properties of $*_X$ under $\mathcal{F}_X$)

$$= \int_S\left|\mathcal{F}_X(f)(\lambda)\right|^2\left(C_\psi^{-1}\int_0^{+\infty}\left|\mathcal{F}_X(\overline{\psi_\delta})(\lambda)\right|^2\pi(\lambda)d\pi(\lambda)\right)$$

(by Fubini-Tonelli)

$$= ||\mathcal{F}_X f||_{\mathbb{L}^2(\hat{X},d\pi)}$$

(using the admissibility condition for ψ given by equation (4.5))

$$= ||f||_{\mathbb{L}^2(X,d\mu)}$$

From (iii) one deduces (iv) by polarization and Fubini-Tonelli.

4.3. **Groups.**

4.3.1. $\mathbb{R}^n$. Consider $\mathbb{R}^n$ with the usual vector sum operation and Lebesgue measure. Consider a one-parameter group of dilations

$$D = \{\exp(tL) : t\in\mathbb{R}\}.$$

It is proven in [36] that a square-integrable function ψ satisfying the admissibility condition exists if and only if $\mathrm{tr}L \neq 0$.

Associated discrete wavelet systems studied in the literature are in the form

$$\{\psi(A^j\cdot - Bk)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d},$$

where A is an expansive matrix (i.e. all its eigenvalues have norm greater than 1) and B is a non-singular matrix,have been widely studied in the literature. The question is whether such a system is a frame. Tight frames of this type have been characterized in full generality in [7] and used to construct new families of anisotropic Hardy spaces in [5].

The space of homogeneous type natural in this setting is $\mathbb{R}^n$ with the quasi-metric ρ defined as follows. Since A is expansive, there exists an open ellipsoid Δ with the property $0 \in \Delta^o \subset A\Delta$. Define $\rho(x,y) = ||x-y||_A$ where

$$||z||_A = \begin{cases} |\det A|^j & \text{if} \quad z \in A^{j+1}\Delta \setminus A^j\Delta \\ 0 & \text{if} \quad z = 0. \end{cases}$$

This "step" quasi-distance ρ can be smoothed by general theorems, see Section 2.

In any dimension and for any expanding matrix A, it is easy to construct associated continuous wavelets, of arbitrary smoothness. We can then choose many tilings $\{Q_{j,k}\}$, for example the affine one

$$Q_{j,k} = A^{-j}([0,1]^n + k),$$

or the quasi-affine one

$$Q_{j,k} = \begin{cases} A^{-j}([0,1]^n + k) & ,j \geq 0 \\ A^{-j}([0,1]^n) + k & ,j < 0 \end{cases}$$

for j,k in $\mathbb{Z}$. Theorem 3.1 then immediately implies the existence of wavelet frames associated to these tilings, in the appropriate $\mathbb{L}^p(\mathbb{R}^n)$ and molecular spaces.

One can also consider radial tilings of $\mathbb{R}^n$, and also weighted $\mathbb{R}^n$ with doubling measures.

4.3.2. $\mathbb{H}^n$. Consider the Heisenberg group of dimension n, defined as the set

$$\mathbb{H}^n = \{(z,t) \in \mathbb{C}^n \times \mathbb{R}\}$$

endowed with the operation

$$(z,t)(z',t') = \left(z+z', t+t'+2\Im\left(z\overline{z'}\right)\right) .$$

Its dilations are defined by $\delta : (z,t) \mapsto (\delta z, \delta^2 t)$ (where we use δ to denote both a positive real number and the associated dilation), and are automorphisms of the group. With the quasi-metric

$$\rho((z,t),(z',t')) = \max\{||z-z'||_{\mathbb{C}^n}, |t-t'|^{1/2}\},$$

which is homogeneous of degree one with respect to the dilations defined above, and the Lebesgue measure, $\mathbb{H}^n$ is a space of homogeneous type. Continuous wavelets, Calderón reproducing formulas and the corresponding admissibility condition are studied, e.g., in [38].

For discrete wavelet systems, we consider the lattice subgroup $\Gamma = \mathbb{Z} \times \mathbb{Z} \times 4\mathbb{Z}$ naturally embedded in $\mathbb{H}^n$. Observe that Γ acts on $\mathbb{H}^n$ and on itself by left translations. Let Q be its fundamental cell, fix $\delta > 0$ and define $Q_{j,k} = \delta^{-j}(k \cdot Q)$, $j \in \mathbb{Z}$, $k \in \Gamma$. Then, for each j, $\mathbb{H}^n$ is tiled by $\{Q_{j,k}\}_{k\in\Gamma}$, and it is natural to consider the generalized affine wavelet systems in the form

$$\psi_{j,k}(\cdot) = \delta^{j(n+1)}\psi(k^{-1}\delta^j \cdot),$$

$j \in \mathbb{Z}$, $k \in \Gamma_j$, where Γ_j is either the affine or quasi-affine lattice.

Continuous wavelets on $\mathbb{H}^n$ have been investigated in [38]. There the two irreducible generalized Hardy spaces of this representation (see [21] for a discussion of the link between generalized Hardy spaces and irreducible subspaces) are decomposed in a countable sequence of invariant subspaces and an admissible wavelet for each subspace is constructed. Among them, we have weighted Bergmann spaces. Theorem 3.1 then yields wavelet frames in each of these spaces.

4.4. **Spheres and hyperboloids.** Our techniques, with few due modifications, are applicable to the continuous wavelets constructed from group actions on the sphere and hyperboloids ([1],[2] and references therein).

5. Some Quasi-Orthogonality Lemmata

Molecules will be our building blocks for analyzing function spaces, discretizing integrals and synthetizing functions. The most basic properties we will need for these molecules are quasi-orthogonality relations. These show an independence of molecules centered at different points and/or having different radii, hence of molecules at different locations and/or scales. These properties will guarantee the quasi-orthogonality of our decompositions and reconstructions, finally yielding the frame properties we seek. We group in this section these results, postponing their proofs in Appendix A.

Lemma 5.1 (Quasi-Orthogonality I). Suppose f, g satisfy only the following decay conditions:

(i) $|f(z)| \leq C_f \frac{r^\gamma}{(r+\rho(z,x))^{\mathrm{n}+\gamma}}$
(ii) $|g(z)| \leq C_g \frac{r^\gamma}{(r+\rho(z,y))^{\mathrm{n}+\gamma}}$

for some x, z in X and for all y in X. Then

$$|< f, g >| \leq 2^{2+\gamma}(C_f + C_g)\frac{r^\gamma}{(r+\rho(x,y))^{\mathrm{n}+\gamma}} \tag{5.1}$$

The special cancellation properties of molecules allow an improvement of the above estimates, yielding also quasi-orthogonality across scales.

Lemma 5.2 (Quasi-Orthogonality II). If f belongs to $\mathcal{M}(x, r_0, \beta, \gamma)$ and g belongs to $\mathcal{M}(y, r_1, \beta, \gamma)$, with $r_0 \leq r_1$ and $\beta < \gamma$, then

$$|< f, g >| \leq C_{\beta,\gamma}||f||_{\mathcal{M}(x,r_0,\beta,\gamma)}||g||_{\mathcal{M}(y,r_1,\beta,\gamma)}\left(\frac{r_0}{r_1}\right)^\beta \frac{r_1^\gamma}{(r_1+\rho(x,y))^{\mathrm{n}+\gamma}} \tag{5.2}$$

Lemma 5.3 (Quasi-Orthogonality III). Let ψ be in $\mathcal{M}(0, r, \beta, \gamma)$, with $\beta \leq \gamma$, $\delta_1 \leq \delta_2$, and $\rho(x_1, x_1') \leq \delta_2 r + \rho(x_1, x_2)$. Then

$$\left|< \psi_{\delta_1,x_1^-} - \psi_{\delta_1,x_1'^-}, \psi_{\delta_2,x_2^-} >\right| \leq C||\psi||_{\mathcal{M}(0,r,\beta,\gamma)}\frac{\rho(x_1,x_1')^\beta(\delta_2 r)^\gamma}{(\delta_2 r+\rho(x_1,x_2))^{\mathrm{n}+\gamma+\beta}} \tag{5.3}$$

Lemma 5.4 (Quasi-Orthogonality IV). Let ψ be in $\mathcal{M}(0, r, \beta, \gamma)$, with $\beta \leq \gamma$, $\delta_1 \geq \delta_2$, and $\rho(x_1, x_1') \leq \delta_1 r + \rho(x_1, x_2)$. Then

$$\left|< \psi_{\delta_1,x_1^-} - \psi_{\delta_1,(x_1')^-}, \psi_{\delta_2,x_2} >\right| \leq C||\psi||_{\mathcal{M}(0,r,\beta,\gamma)}\frac{\rho(x_1,x_1')^\beta(\delta_1 r)^\gamma}{(\delta_1 r+\rho(x_1,x_2))^{\mathrm{n}+\gamma+\beta}} \tag{5.4}$$

We conclude this section with two integration formulae and two simple properties of molecules.

Fact 5.5. We have the following

$$\int_{B(0,r)} \frac{d\mu(x)}{\rho(x,0)^{\mathrm{n}-\gamma}} = Cr^\gamma \ , \qquad \int_{B(0,r)^C} \frac{d\mu(x)}{\rho(x,0)^{\mathrm{n}+\gamma}} = Cr^{-\gamma}$$

Proof. Divide the space into dyadic annuli. □

Fact 5.6. For any $\alpha > -\gamma$ we have

$$\int_0^{+\infty} \frac{(\delta r)^\gamma}{(\delta r+\rho(x,y))^{\mathrm{n}+\gamma+\alpha}}\frac{d\delta}{\delta} \leq 2\rho(x,y)^{-\alpha} \tag{5.5}$$

In particular, $||f||_{\mathbb{L}^1(X,\mu)} \leq 2$ for any molecule.

Fact 5.7. There exists C, independent of r,γ, such that

$$\int_G \frac{r^\gamma}{(r+\rho(x,y))^{\mathrm{n}+\gamma}}|f(y)|d\mu(y) \le C\,(Mf)\,(x)$$

where Mf is the Hardy-Littlewood maximal function defined by

$$(Mf)\,(x) = \sup_{r>0} \frac{1}{r^{\mathrm{n}}} \int_{\{y:\rho(x,y)\le r\}} |f(y)|d\mu(y)\ .$$

Proof. Divide the space into dyadic annuli. □

6. $\mathbb{L}^2$-boundedness of the Riemann sum operator

The goal of this section is to show that $S^{(a,b)}$ is bounded in $\mathbb{L}^2(X,\mu)$, which implies that $S^{(a,b)}$ is indeed well-defined. We will use estimates based on generalized Littlewood-Paley decomposition that we now introduce (related estimates in the standard wavelet settings are in [8],[11],[19],[26],[25],[27],[28],[29],[31],[40]).

By duality, it is enough to prove that $\{\psi^{(a,b)}_{j,k}\}_{j,k}$ is a Bessel sequence, i.e.

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}_j} \left|< f,\psi^{(a,b)}_{j,k} >\right|^2 \le C||f||_2^2\ .$$

We need the following definition:

Definition 6.1 (Han). A sequence of operators $\{D_j\}_{j\in\mathbb{Z}}$ is called a sequence of (β,γ)-MultiScale Operators if the kernels $\{D_j(x,y)\}$ are such that there exist constants $a>1$ and $C_a>0$ such that the following estimates are satisfied:

(i) Decay condition

$$|D_kD_j(x,y)| \le C_a a^{-|k-j|\gamma} \frac{a^{-(k\wedge j)\gamma}}{\left(a^{-(k\wedge j)}+\rho(x,y)\right)^{\mathrm{n}+\gamma}} \tag{6.1}$$

(ii) Smoothness condition in the first variable:

$$|D_kD_j(x,y)-D_kD_j(x',y)| \le C_a \left(\frac{\rho(x,x')}{a^{-j}+\rho(x,y)}\right)^\beta \frac{a^{j\gamma}}{\left(a^{-j}+\rho(x,y)\right)^{\mathrm{n}+\gamma}} \tag{6.2}$$

for $\rho(x,x') \le \left(a^{-j}+\rho(x,y)\right)$.

(iii) If $j\ge k$, a second smoothness condition in the first variable:

$$|D_kD_j(x,y)-D_kD_j(x',y)| \le C_a \left(\frac{\rho(x,x')}{a^{-k}+\rho(x,y)}\right)^\beta \frac{a^{k\gamma}}{\left(a^{-k}+\rho(x,y)\right)^{\mathrm{n}+\gamma}} \tag{6.3}$$

for $\rho(x,x') \le \left(a^{-k}+\rho(x,y)\right)$.

(iv) Double-smoothness condition:

$$\begin{aligned} &|D_kD_j(x,y)-D_kD_j(x',y)-D_kD_j(x,y')+D_kD_j(x',y')| \\ &\le C_a \left(\frac{\rho(x,x')}{a^{-(k\wedge j)}+\rho(x,y)}\right)^\beta \left(\frac{\rho(y,y')}{a^{-(k\wedge j)}+\rho(x,y)}\right)^\beta \frac{a^{-(k\wedge j)\gamma}}{\left(a^{-(k\wedge j)}+\rho(x,y)\right)^{\mathrm{n}+\gamma}} \end{aligned} \tag{6.4}$$

for $\rho(x,x') \le a^{-(k\wedge j)}+\rho(x,y)$, $\rho(y,y') \le a^{-(k\wedge j)}+\rho(x,y)$.

(v) Vanishing moment in each variable:

$$\int D(x,y)d\mu(y) = \int D(x,y)d\mu(x) = 0. \tag{6.5}$$

Theorem 6.1 ([26]). If $\{D_j\}_{j\in\mathbb{Z}}$ is a sequence of (β,γ)-Multi-Scale Operators, then there exist sequences of operators $\{\tilde{D}_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{\tilde{D}}_j\}_{j\in\mathbb{Z}}$ such that for $f \in \mathcal{M}(x_0, r_0, \beta, \gamma)$

$$f = \sum_{j\in\mathbb{Z}} \tilde{D}_j D_j(f) = \sum_{j\in\mathbb{Z}} D_j \tilde{\tilde{D}}_j(f)$$

where the series converges in the norm of $\mathcal{M}(x_0, r_0, \beta', \gamma')$ for any $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$, and in the spaces $\mathbb{L}^p(X,\mu)$, for $1 < p < \infty$. Moreover, the kernels of $\tilde{D}_j$ satisfy conditions (ii) and (iii) in Definition 6.1 with β,γ replaced by β',γ', for any $0 < \beta' < \beta, 0 < \gamma' < \gamma$, and $\int D_j(x,y)dy = 0$; the kernels of $\tilde{\tilde{D}}_j$ satisfy the corresponding conditions in the y-variable.

After having seen the quasi-orthogonality relations of molecules, it is easy to prove that the operators

$$\left(D_j(f)\right)(\cdot) =< f, \phi_{j,\cdot} >_{\mathbb{L}^2(X,\mu)},$$

for ϕ in $\mathcal{M}(0,r,\beta,\gamma)$, define a collection of (β,γ)-Multi-Scale Operators. In fact this definition implies

$$\left(D_k D_j\right)(x,z) =< \phi_{j,x}, \phi_{k,z} >_{\mathbb{L}^2(X,\mu)}$$

and the quasi-orthogonality estimates across translations and dilations yield the desired conditions. Also, observe the relations

$$\left(D_j^* g\right)(\cdot) =< g, \overline{\phi}_{j,\cdot} >_{\mathbb{L}^2(X,\mu)},$$

which follow from our assumptions (T2),(T3),(D2). To see that the double smoothness condition (6.4) is satisfied, it is enough to repeat the proof of 5.3 by using the smoothness condition on the second factor of the inner product.

We can now estimate $S^{(a,b)}f$ by first decomposing f in a semi-discrete fashion with the D_j's associated with ϕ, as suggested by the above Theorem, and then compare this semi-discrete decomposition with the fully discrete one, associated ψ: if (W2) holds

$$\sum_j \sum_{k\in\mathcal{K}(j)} \left|< f, \lambda_{j,k}^{(a,b)} \psi_{j,k}^{(a,b)} >\right|^2 = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \left|< \sum_{i\in\mathbb{Z}} D_i \tilde{\tilde{D}}_i f, \lambda_{j,k}^{(a,b)} \psi_{j,k}^{(a,b)} >\right|^2$$

$$= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \left|\sum_{i\in\mathbb{Z}} \lambda_{j,k}^{(a,b)} < \tilde{\tilde{D}}_i f(\cdot), < \psi_{j,k}^{(a,b)}, \overline{\tilde{\phi}}_{i,\cdot} >>\right|^2$$

$$\leq C \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \left(\sum_i a^{-|i-j|\gamma} \lambda_{j,k}^{(a,b)} \int_X \frac{a^{-(i\wedge j)\gamma}}{\left(a^{-(i\wedge j)} + \rho(x, z_{j,k}^{(a,b)})\right)^{n+\gamma}} |\tilde{\tilde{D}}_i(f)|(x) d\mu(x) \right)^2$$

$$\leq C \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \left(\lambda_{j,k}^{(a,b)}\right)^2 \left(\sum_{i\in\mathbb{Z}} a^{-|i-j|\gamma} \right)$$

$$\cdot \left(\sum_{i\in\mathbb{Z}} a^{-|i-j|\gamma} \left(\int_X \frac{a^{-(i\wedge j)\gamma}}{\left(a^{-(i\wedge j)} + \rho(x, z_{j,k}^{(a,b)})\right)^{n+\gamma}} |\tilde{\tilde{D}}_i(f)|(x) d\mu(x) \right)^2 \right)$$

(by Cauchy-Schwarz inequality)

$$\leq C \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} a^{-|i-j|\gamma} \left(\lambda_{j,k}^{(a,b)}\right)^2 \mu(Q_{j,k})^{-1} \int_X \left| M(\tilde{\tilde{D}}_i(f))(z)\right|^2 d\mu(z)$$

(by our choice of $\lambda_{j,k}^{(a,b)}$ and applying Lemma 5.7)

$$\leq C\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} a^{-|i-j|\gamma}||\tilde{\tilde{D}}_i(f)||_2^2$$

$$\leq C\sum_{i\in\mathbb{Z}}||\tilde{\tilde{D}}_i(f)||_2^2 \leq C||f||_2^2$$

by invoking the $\mathbb{L}^2$-boundedness of the maximal function and Littlewood-Paley theory for spaces of homogeneous type [26]. The constant C in these equalities can be chosen to be independent of f so, by a density argument, the above estimate holds for $f \in \mathbb{L}^2(X,\mu)$, which is what we wanted to prove. The proof in the case (W1) is altogether similar.

7. Discretization

We want to measure the difference between the continuous reproducing formula and our tentative Riemann sums. This difference is an operator bounded on many function spaces, and whose norm can be controlled, and in fact made arbitrarily small, by choosing appropriately the discretization parameters a, b.

In detail:

$$\left(R^{(a,b)}f\right)(x) =$$

$$= \int_0^{+\infty}\int_X < f, \psi_{\delta,y} > \psi_{\delta,y}(x)d\mu(y)\frac{d\delta}{\delta}$$

$$- \ln a\sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)} < f, \mu(Q_{j,k}^{(a,b)})^{\frac{1}{2}}\psi_{j,k}^{(a,b)} > \mu(Q_{j,k}^{(a,b)})^{\frac{1}{2}}\psi_{j,k}^{(a,b)}(x)$$

$$= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)}\int_{a^{-j}}^{a^{-j+1}}\int_{Q_{j,k}^{(a,b)}}\{\psi_{\delta,y}(x) - \psi_{\delta,k}(x)\} < f, \psi_{\delta,y} > d\mu(y)\frac{d\delta}{\delta}$$

$$+ \sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)}\int_{a^{-j}}^{a^{-j+1}}\int_{Q_{j,k}^{(a,b)}}\{< f, \psi_{\delta,y} > - < f, \psi_{\delta,k} >\}\psi_{\delta,k}(x)d\mu(y)\frac{d\delta}{\delta}$$

$$+ \sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)}\int_{a^{-j}}^{a^{-j+1}}\int_{Q_{j,k}^{(a,b)}}\left\{\psi_{\delta,k}(x) - \psi_{j,k}^{(a,b)}(x)\right\} < f, \psi_{\delta,k} > d\mu(y)\frac{d\delta}{\delta}$$

$$+ \sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)}\int_{a^{-j}}^{a^{-j+1}}\int_{Q_{j,k}^{(a,b)}}\left\{< f, \psi_{\delta,k} > - < f, \psi_{j,k}^{(a,b)} >\right\}\psi_{j,k}^{(a,b)}(x)d\mu(y)\frac{d\delta}{\delta}$$

$$= \left(R_1^{(a,b)} + R_2^{(a,b)} + R_3^{(a,b)} + R_4^{(a,b)}\right)(f)$$

We shall now prove that the four operators $\{R_j^{(a,b)}\}_{j\in\{1\ldots 4\}}$ are nice Calderón-Zygmund singular integral operators :

Lemma 7.1. R is a Calderón-Zygmund singular integral operator which is bounded on $\mathbb{L}^2(G)$. Moreover, for any $0 < \beta' < \beta \leq \gamma$ there exists C, independent of a and b, such that each $R_j^{(a,b)}$, for any $j \in \{1,\ldots,4\}$, satisfies the following estimates:

$$|R_j^{(a,b)}(x,y)| \leq C||\psi||^2\left((a-1)^{\mathrm{d}} \vee b^\beta\right)\rho(x,y)^{-\mathrm{n}} \tag{7.1}$$

$$|R_j^{(a,b)}(x',y) - R_j^{(a,b)}(x,y)| \leq C||\psi||^2\rho(x,x')^{\beta'}\rho(x,y)^{-\mathrm{n}-\beta'} \tag{7.2}$$

for $\rho(x,x') \leq \frac{1}{2}\rho(x,y)$

$$|R_j^{(a,b)}(x,y) - R_j^{(a,b)}(x,y')| \leq C||\psi||^2\rho(y,y')^{\beta'}\rho(x,y)^{-\mathrm{n}-\beta'} \tag{7.3}$$

for $\rho(y,y') \le \frac{1}{2}\rho(x,y)$

$$\begin{aligned}\left|\left(R_j^{(a,b)}(x,y) - R_j^{(a,b)}(x',y)\right) - \left(R^{(a,b)}(x,y) - R^{(a,b)}(x,y')\right)\right| \\ \le C||\psi||^2 \rho(x,x')^{\beta'}\rho(y,y')^{\beta'}\rho(x,y)^{-\mathrm{n}-\beta'}\end{aligned} \tag{7.4}$$

for $\rho(x,x') \le \frac{1}{4}\rho(x,y)$ and $\rho(y,y') \le \frac{1}{4}\rho(x,y)$

$$\left|< R_j^{(a,b)}, f >_{\mathbb{L}^2(G\times G)}\right| \le ||\psi||^2 C\left(C_{\mathrm{d},\beta,\gamma}(a-1)^{\mathrm{d}} \vee b^{\beta}\right) r^{\mathrm{n}} \tag{7.5}$$

for all $f \in \mathcal{C}_0^\eta(G\times G)$, $0<\eta<\beta$, with supp.$f \subset B(x_1,r)\times B(y_1,r)$, $x_1,y_1 \in G$, $||f||_\infty \le 1$, $||f(\cdot,y)||_\eta \le r^{-\eta}$, $||f(x,\cdot)||_\eta \le r^{-\eta}$ for all $x,y\in G$.

The (rather technical) proof can be found in Appendix A.

Observe that the estimates in Lemma 7.1 still do not allow a complete control of $||R_j^{(a,b)}||_{CZ}$, since a,b do not appear in estimates (7.2)-(7.4). The idea is to use a geometric mean argument to combine (7.1) with (7.2)-(7.4). For example, for (7.2), using the assumption $\rho(y,y') \le \frac{1}{2}\rho(x,y)$ and the (quasi-)triangle inequality, we obtain

$$\left|R_j^{(a,b)}(x,y) - R_j^{(a,b)}(x,y')\right| \le C\left((a-1)^{\mathrm{d}} \vee b^{\beta}\right)\rho(x,y)^{-\mathrm{n}} . \tag{7.6}$$

By using estimate (7.2) for $\left|R_j^{(a,b)}(x,y) - R_j^{(a,b)}(x,y')\right|^p$ and estimate (7.6) for $\left|R_j^{(a,b)}(x,y) - R_j^{(a,b)}(x,y')\right|^{1-p}$ and choosing p appropriately, we obtain that for any β' with $0<\beta'<\beta<\gamma$ there exists $\mathrm{d}', \beta''>0$ and $C>0$ so that

$$\left|R_j^{(a,b)}(x,y) - R_j^{(a,b)}(x,y')\right| \le C\left((a-1)^{\mathrm{d}'} \vee b^{\beta''}\right)\rho(y,y')^{\beta'}\rho(x,y)^{-\mathrm{n}-\beta'}$$

for $\rho(y,y') \le \frac{1}{2}\rho(x,y)$. Similar estimates can be obtained for (7.3) and (7.4). Then we can control the norm of $R_j^{(a,b)}$ by choosing a,b: there exists a constant $C(a,b)$, $C(a,b)\to 0^+$ as $a\to 1^+$ and $b\to 0$, such that

$$||R^{(a,b)}f||_{\mathcal{M}(\beta',\gamma)} \le C(a,b)||f||_{\mathcal{M}(\beta',\gamma)}.$$

Thus, we can fix a,b so that $C(a,b)<1$, the operator $R^{(a,b)} = I - S^{(a,b)}$ has norm less than 1, and hence $S^{(a,b)}$ is invertible on $\mathcal{M}(\beta',\gamma)$. It is not hard to see that $R1 = R^*1 = 0$ in BMO (since BMO is a quotient modulo constant functions), so the T1 Theorem can be applied.

Finally, to prove that $\{\psi_{j,k}^{(a,b)}\}$ is a frame, we use our assumption that the continuous wavelet transform operator

$$f \mapsto \int_X f * \tilde{\psi}_\delta * \psi_\delta$$

is the identity on $\mathbb{L}^2(X,\mu)$ (in fact, we only need that, up to a renormalization of ψ, this operator is less than 1 far from the identity, in the $\mathbb{L}^2(X,\mu_X)$ sense!). But we have just seen that the Riemann sum operator $S^{(a,b)}$ can be made arbitrarily close, in the $\mathbb{L}^2(X,\mu_X)$ sense, to this continuous wavelet transform operator, by choosing a,b appropriately. So, for appropriate a,b's, $S^{(a,b)}$ is invertible on $\mathbb{L}^2(X,\mu_X)$.

The standard argument

$$\begin{aligned}||S^{(a,b)}f||_2 &= \sup_{||g||_2=1}\left|< S^{(a,b)}f, g >\right| \\ &\le \sup_{||g||_2=1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)}\left|< f, \psi_{j,k}^{(a,b)} >\right|\left|< g, \psi_{j,k}^{(a,b)} >\right|\end{aligned}$$

$$\leq \sup_{||g||_2=1} \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \left| < f, \psi^{(a,b)}_{j,k} > \right|^2 \right)^{\frac{1}{2}} \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \left| < g, \psi^{(a,b)}_{j,k} > \right|^2 \right)^{\frac{1}{2}}$$

$$\leq ||S^{(a,b)}||_{\mathbb{L}^2(X,\mu)} \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \left| < f, \psi^{(a,b)}_{j,k} > \right|^2 \right)^{\frac{1}{2}} .$$

gives, since $S^{(a,b)}$ is invertible,

$$||f||_2 \leq C||S^{(a,b)} f||_2 \leq C' \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \left| < f, \psi^{(a,b)}_{j,k} > \right|^2 \right)^{\frac{1}{2}} \leq C''||f||_2 \ ,$$

so $\{\psi_{j,k}\}_{j,k}$ is a frame.

Finally, to get our completely discretized reproduction formula, we define $\phi_{j,k} = \left(S^{(a,b)}\right)^{-1} \psi^{(a,b)}_{j,k}$. Since $S^{(a,b)}$ was bounded and invertible on $\mathcal{M}(\beta', \gamma)$, $\phi^{(a,b)}_{j,k}$ is a molecule, and

$$\begin{aligned} f &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} < f, \phi^{(a,b)}_{j,k} > \psi^{(a,b)}_{j,k} \\ &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} < f, \psi^{(a,b)}_{j,k} > \phi^{(a,b)}_{j,k} \end{aligned} \tag{7.7}$$

with convergence in $\mathbb{L}^p(G)$, for $1 < p < \infty$ and $\mathcal{M}(\beta', \gamma)$. The convergence in $\mathbb{L}^p(G)$ follows from the following argument: first of all,

$$f = \sum_{|j|<J} \sum_{|k|<K} < f, \psi_{j,k} > \phi_{j,k} + S^{-1} \left(\sum_{|j|>J \text{or} |k|>K} < f, \phi_{j,k} > \psi_{j,k} \right) .$$

Hence, by the boundedness of $S^{(a,b)}$ on $\mathbb{L}^p(G)$, it is enough to show that

$$\lim_{J,K\to+\infty} \left\| \sum_{|j|>J \text{ or } |k|>K} < f, \psi_{j,k} > \phi_{j,k} \right\|_p = 0.$$

By duality and Hahn-Banach, if q is the conjugate exponent of p

$$\left\| \sum_{|j|>J \text{or} |k|>K} < f, \psi_{j,k} > \phi_{j,k} \right\|_p = \sup_{||g||_q=1} \left| \sum_{|j|>J \text{ or } |k|>K} << f, \psi_{j,k} > \phi_{j,k}, g > \right|$$

$$\leq \sup_{||g||_q=1} \sum_{|j|>J \text{or} |k|>K} |< f, \psi_{j,k} >| \, |\phi_{j,k}, g >|$$

$$\leq C \sum_{|j|>J} \sum_{k\in\mathcal{K}(j)} \int_{Q^{(a,b)}_{j,k}} \left(\sum_i a^{-|i-j|\gamma'} M(\tilde{\tilde{D}}_i f)(z) \right)$$

$$\cdot \left(\sum_i a^{-|i-j|\gamma'} M(\tilde{\tilde{D}}_i g)(z) \right)$$

(as in the proof of the boundedness of $S^{(a,b)}$, here z is any point in $Q^{(a,b)}_{j,k}$)

$$\leq C \int_G \left(\sum_{|j|>J} \left(\sum_i a^{-|i-j|\gamma'} M(\tilde{\tilde{D}}_i f)(z) \right)^2 \right)^{\frac{1}{2}}$$

$$\cdot \left(\sum_{|j|>J} \left(\sum_{i} a^{-|i-j|\gamma'} M(\tilde{\tilde{D}}_i g)(z) \right)^2 \right)^{\frac{1}{2}}$$

(by Hölder inequality)

$$\leq Ca^{-J/2}||f||_p||g||_q + C \left\| \left(\sum_{|i|>J/2} \left(M\left(\tilde{\tilde{D}}_i f\right)\right)^2 \right)^{\frac{1}{2}} \right\|_p$$

$$\cdot \left\| \left(\sum_{|i|>J/2} \left(M\left(\tilde{\tilde{D}}_i g\right)\right)^2 \right)^{\frac{1}{2}} \right\|_q$$

by Littlewood-Paley theory, and this goes to 0 as $J \to +\infty$. The convergence in $\mathcal{M}(\beta', \gamma)$ is proved similarly, with an appeal to the Lebesgue dominated convergence theorem.

Finally, the claim that $\phi_{j,k}$ is a dilated version of $\phi_{0,k}$ follows immediately from the fact the S, hence S^{-1}, commutes with dilations. This finished the proof of Theorem 3.1.

8. Appendix A. Proofs of the Quasi-Orthogonality Lemmata

We assume that C_ρ, the constant in the quasi-triangle inequality (2.1), is equal to 1. A look at the proofs will convince the reader that we do so without loss of generality.

Proof of Lemma 5.1. We have

$$\begin{aligned}
| < f, g > | \leq & \int_{\rho(x,z) \geq \frac{1}{2}(\rho(x,y)-r)} C_f \frac{r^\gamma}{(r + \rho(z,x))^{\mathrm{n}+\gamma}} |g(z)| d\mu(z) \\
& + \int_{\rho(x,z) \leq \frac{1}{2}(\rho(x,y)-r)} |f(z)| C_g \frac{r^\gamma}{(r + \rho(z,y))^{\mathrm{n}+\gamma}} d\mu(z) \\
\leq & \int_{\rho(x,z) \geq \frac{1}{2}(\rho(x,y)-r)} C_f \frac{r^\gamma}{\left(\frac{r}{2} + \frac{\rho(x,y)}{2}\right)^{\mathrm{n}+\gamma}} |g(z)| d\mu(z) \\
& + \int_{\rho(x,z) \leq \frac{1}{2}(\rho(x,y)-r)} |f(z)| C_g \frac{r^\gamma}{\left(\frac{3}{2}r + \frac{\rho(x,y)}{2}\right)^{\mathrm{n}+\gamma}} d\mu(z) \\
\leq & 2^{\mathrm{n}+\gamma} \left(C_f ||g||_{\mathbb{L}^1(X,\mu)} + ||f||_{\mathbb{L}^1(X,\mu)} C_g \right) \frac{r^\gamma}{(r + \rho(x,y))^{\mathrm{n}+\gamma}}.
\end{aligned}$$

□

Proof of Lemma 5.2. For ease of notation let us write $||f||$ for $||f||_{\mathcal{M}(x,r_0,\beta,\gamma)}$ and $||g||$ for $||g||_{\mathcal{M}(y,r_1,\beta,\gamma)}$. We partition the space into the three sets

$$X_1 = \left\{ z \in X : \rho(z,x) \leq \frac{r_1}{2} + \rho(z,y) \right\}$$

$$X_2 = \left\{ z \in X : \rho(z,x) < \frac{1}{2}(\rho(x,y) + r_1), \rho(z,x) > \frac{r_1}{2} + \rho(z,y) \right\}$$

$$X_3 = \left\{ z \in X : \rho(z,x) \geq \frac{1}{2}(\rho(x,y) + r_1), \rho(z,x) > \frac{r_1}{2} + \rho(z,y) \right\}$$

and we estimate:

$$
\begin{aligned}
|<f,g>| \le\ & ||f||||g|| \int_{X_1} \frac{r_0^\gamma}{(r_0+\rho(x,z))^{n+\gamma}} \frac{\rho(x,z)^\beta r_1^\gamma}{(r_1+\rho(y,z))^{n+\gamma+\beta}} d\mu(z) + \\
& + ||f||||g|| \int_{X_2} \frac{r_0^{\gamma-\beta} r_0^\beta}{(r_0+\rho(x,z))^{n+\gamma-\beta} \left(\frac{r_1}{2}\right)^\beta} \frac{r_1^\gamma}{(r_1+\rho(y,z))^{n+\gamma}} d\mu(z) \\
& + ||f||||g|| \int_{X_3} \frac{r_0^{\gamma-\beta} r_0^\beta}{(r_0+\rho(x,z))^{n+\gamma-\beta} \left(\frac{r_1}{2}\right)^\beta} \frac{r_1^\gamma}{(r_1+\rho(y,z))^{n+\gamma}} d\mu(z) \\
\le\ & ||f||||g|| \int_{X_1} \frac{r_0^{\gamma-\beta} r_0^\beta}{(r_0+\rho(x,z))^{n+\gamma}} \frac{(r_0+\rho(x,z))^\beta r_1^\gamma}{(r_1+\rho(x,y))^{n+\gamma} r_1^\beta} d\mu(z)
\end{aligned}
$$

(since on X_1 we have $\rho(x,y) \le \rho(z,x)+\rho(z,y) \le \frac{r_1}{2} + 2\rho(z,y)$, so $r_1 + \rho(z,y) \ge \frac{3}{4} r_1 + \frac{1}{2}\rho(x,y)$)

$$
+ ||f||||g|| \int_{X_2} \left(\frac{r_0}{r_1}\right)^\beta 2^\beta \frac{r_0^{\gamma-\beta} r_0^\beta}{(r_0+\rho(x,z))^{n+\gamma-\beta}} \frac{r_1^\gamma}{\left(\frac{r_1}{2} + \frac{\rho(x,y)}{2}\right)^{n+\gamma}} d\mu(z)
$$

(since on X_2 we have $\rho(x,y) \le \rho(x,z) + \rho(y,z) \le \frac{1}{2}\rho(x,y) + \frac{r_1}{2} + \rho(y,z)$, hence $\rho(x,y) \le 2\rho(y,z) + 2r_1$)

$$
+ ||f||||g|| \int_{X_3} \left(\frac{r_0}{r_1}\right)^\beta 2^\beta \frac{r_0^{\gamma-\beta} r_0^\beta}{(r_0+\rho(x,z))^{n+\gamma-\beta}} \frac{r_1^\gamma}{\left(\frac{r_1}{2} + \frac{\rho(x,y)}{2}\right)^{n+\gamma}} d\mu(z)
$$

(since on X_3 we have $\rho(y,z) > \rho(x,z) - \frac{r_1}{2} > \frac{1}{2}\rho(x,y)$)

$$
\begin{aligned}
&\le ||f||||g|| \left(2^{n+\gamma} ||f||_{\mathbb{L}^1(X,\mu)} + 2^{n+\beta+\gamma} 2 + 2^{n+\beta+\gamma} 4\right) \left(\frac{r_0}{r_1}\right)^\beta \frac{r_1^\gamma}{(r_1+\rho(x,y))^{n+\gamma}} \\
&\le C_{\beta,\gamma} ||f||||g|| \left(\frac{r_0}{r_1}\right)^\beta \frac{r_1^\gamma}{(r_1+\rho(x,y))^{n+\gamma}}
\end{aligned}
$$

□

Proof of Lemma 5.3. Again let us write $||\psi||$ instead of $||\psi||_{\mathcal{M}(0,r,\beta,\gamma)}$. We divide the space into the three sets

$$
\begin{aligned}
X_1 &= \{z \in X : \rho(x_1,x_1') \le \delta_2 r + \rho(x_1,x_2) \le \delta_1 r + \rho(x_1,z)\} \\
X_2 &= \{z \in X : \rho(x_1,x_1') \le \delta_1 r + \rho(x_1,z) \le \delta_2 r + \rho(x_1,x_2)\} \\
X_3 &= \{z \in X : \rho(x_1,x_1') > \delta_1 r + \rho(x_1,z)\}
\end{aligned}
$$

and estimate:

$$
\begin{aligned}
&|<\psi_{\delta_1,x_1} - \psi_{\delta_1,x_1'}, \psi_{\delta_2,x_2}>| \le \\
&\quad \le ||\psi||^2 \int_{X_1} \frac{\rho(x_1,x_1')^\beta (\delta_1 r)^\gamma}{(\delta_1 r + \rho(x_1,z))^{n+\gamma+\beta}} \\
&\qquad \left(\frac{(\delta_2 r)^\gamma}{(\delta_2 r + \rho(x_2,z))^{n+\gamma}} + \frac{(\delta_2 r)^\gamma}{(\delta_2 r + \rho(x_2,x_1))^{n+\gamma}} \right) d\mu(z) \\
&\quad + ||\psi||^2 \int_{X_2} \frac{\rho(x_1,x_1')^\beta (\delta_1 r)^\gamma}{(\delta_1 r + \rho(x_1,z))^{n+\gamma+\beta}} \frac{\rho(x_1,z)^\beta (\delta_2 r)^\gamma}{(\delta_2 r + \rho(x_2,x_1))^{n+\gamma+\beta}} d\mu(z) \\
&\quad + ||\psi||^2 \int_{X_3} \left(\frac{(\delta_1 r)^\gamma}{(\delta_1 r + \rho(x_1,z))^{n+\gamma}} + \frac{(\delta_1 r)^\gamma}{(\delta_1 r + \rho(x_1',z))^{n+\gamma}} \right)
\end{aligned}
$$

$$\frac{\rho(x_1,z)^\beta(\delta_2 r)^\gamma}{(\delta_2 r+\rho(x_2,x_1))^{\mathrm{n}+\gamma+\beta}}d\mu(z)$$
$$\leq ||\psi||^2 C_{\beta,\gamma}\frac{\rho(x_1,x_1')^\beta(\delta_2 r)^\gamma}{(\delta_2 r+\rho(x,x_2))^{\mathrm{n}+\gamma+\beta}}$$

(by Quasi-Orthogonality II, using $\delta_1\leq\delta_2$, and by the integrability of molecules)

$$+\,||\psi||^2\frac{\rho(x_1,x_1')^\beta}{(\delta_2 r+\rho(x,x_2))^{\mathrm{n}+\gamma+\beta}}\int_X\frac{\rho(x,z)^\beta(\delta_2 r\delta_1 r)^\gamma}{(\delta_1 r+\rho(x,z))^{\mathrm{n}+\gamma+\beta}}$$
$$+\,||\psi||^2 4\frac{\rho(x_1,x_1')^\beta(\delta_2 r)^\gamma}{(\delta_2 r+\rho(x,x_2))^{\mathrm{n}+\gamma+\beta}}$$
$$\leq C_{\beta,\gamma}||\psi||^2\frac{\rho(x_1,x_1')^\beta(\delta_2 r)^\gamma}{(\delta_2 r+\rho(x_1,x_2))^{\mathrm{n}+\gamma+\beta}}$$

(by the integrability of molecules)

□

Proof of Lemma 5.4. Similar to the proof of Quasi-Orthogonality IV. □

Proof of Fact 5.6. This is an easy application of the integration formulae.

$$\int_0^{+\infty}\frac{(\delta r)^\gamma}{(\delta r+\rho(x,y))^{\mathrm{n}+\gamma+\alpha}}\frac{d\delta}{\delta}\leq\int_0^{\frac{\rho(x,y)}{r}}\frac{(\delta r)^\gamma}{\rho(x,y)^{\mathrm{n}+\gamma+\alpha}}\frac{d\delta}{\delta}+\int_{\frac{\rho(x,y)}{r}}^{+\infty}\frac{(\delta r)^\gamma}{(\delta r)^{\mathrm{n}+\gamma+\alpha}}\frac{d\delta}{\delta}$$
$$\leq 2\rho(x,y)^{-\mathrm{n}-\alpha}$$

□

9. Appendix B. Proof of Lemma 7.1

In all that follows, we will write $||\psi||$ instead of $||\psi||_{\mathcal{M}(0,r,\beta,\gamma)}$. We also keep assuming that C_ρ is equal to 1, as in the previous section.

9.1. **Estimates for R_1.** If the assumptions and definitions of (W1) hold, the kernel of the operator $R_1^{(a,b)}$ is

$$R_1(x,z)=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)}\int_{a^{-j}}^{a^{-j+1}}\int_{Q_{j,k}^{(a,b)}}\{T_y\psi_\delta(x)-T_{z_{j,k}}\psi_\delta(x)\}\,T_y\psi_\delta(z)d\mu(y)\frac{d\delta}{\delta}$$
$$=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)}\int_{a^{-j}}^{a^{-j+1}}\int_{Q_{j,k}^{(a,b)}}\{T_x\psi_\delta(y)-T_x\psi_\delta(z_{j,k})\}\,T_y\psi_\delta(z)d\mu(y)\frac{d\delta}{\delta}.$$

We prove the decay condition:

$$|R_1(x,z)|\leq\int_0^{+\infty}\int_X||\psi||^2\frac{(C_2 b\delta r)^\beta(\delta r)^\gamma}{(\delta r+\rho(x^-,y))^{\mathrm{n}+\beta+\gamma}}\frac{(\delta r)^\gamma}{(\delta r+\rho(y,z^-))^{\mathrm{n}+\gamma}}d\mu(y)\frac{d\delta}{\delta}$$

(since $T_x\psi_\delta$ is in $\mathcal{M}(x,\delta r,\beta,\gamma)$ and $\rho(y,z_{j,k})\leq C_2\delta br\leq\delta r+\rho(y,x^-)$ uniformly for all small enough b)

$$\leq\int_0^{+\infty}\int_X||\psi||^2 b^\beta\frac{(\delta r)^\gamma}{(\delta r+\rho(x^-,y))^{\mathrm{n}+\gamma}}\frac{(\delta r)^\gamma}{(\delta r+\rho(y,z^-))^{\mathrm{n}+\gamma}}d\mu(y)\frac{d\delta}{\delta}$$

$$\leq \int_0^{+\infty} \int_X ||\psi||^2 b^\beta \frac{(\delta r)^\gamma}{(\delta r + \rho(x^-, z^-))^{\mathrm{n}+\gamma}} \frac{d\delta}{\delta}$$

(by Quasi-Orthogonality I)

$$\leq 2b^\beta ||\psi||^2 \rho(x^-, z^-)^{-\mathrm{n}}$$

(by Lemma 5.6)

Othwerwise, if the assumptions and definitions of (W2) hold, the kernel of the operator $R_1^{(a,b)}$ is

$$R_1(x,z) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \int_{a^{-j}}^{a^{-j+1}} \int_{Q_{j,k}^{(a,b)}} \left\{ T_{y^-}\psi_\delta(x) - T_{z_{j,k}^-}\psi_\delta(x) \right\} T_{y^-}\psi_\delta(z) d\mu(y) \frac{d\delta}{\delta}$$
$$= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \int_{a^{-j}}^{a^{-j+1}} \int_{Q_{j,k}^{(a,b)}} \left\{ T_{x^-}\tilde{\psi}_\delta(y) - T_{x^-}\tilde{\psi}_\delta(z_{j,k}) \right\} T_{y^-}\psi_\delta(z) d\mu(y) \frac{d\delta}{\delta},$$

and we can estimate as above, using the hypothesis that $\tilde{\psi}$ is in $\mathcal{M}(0, r, \beta, \gamma)$, but without need of the involution invariance of the metric ρ.

In the rest of this section, we will show the relevant estimates only for the case (W1), since those for the case (W2) are similar. We now turn to the smoothness condition in x: for $\rho(x, x') \leq \frac{\rho(x,z)}{2}$ we estimate

$$|R_1(x,z) - R(x',z)| =$$
$$\left| \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \int_{a^{-j}}^{a^{-j+1}} \int_{Q_{j,k}^{(a,b)}} (T_x\psi_\delta(y) - T_x\psi_\delta(z_{j,k}) - T_{x'}\psi_\delta(y) + T_{x'}\psi_\delta(z_{j,k}))\, T_y\psi_\delta(z) d\mu(y) \frac{d\delta}{\delta} \right|$$
$$\leq \int_0^{+\infty} |< T_x\psi_\delta - T_{x'}\psi_\delta, T_z\psi_\delta >| \frac{d\delta}{\delta} +$$
$$\sum_{j\in\mathbb{Z}} \int_{a^{-j}}^{a^{-j+1}} \sum_{\substack{k:\rho(x^-,x'^-)\leq\delta r+\rho(x^-,z_{j,k}) \text{ or} \\ \rho(x^-,x'^-)\leq\delta r+\rho(x'^-,z_{j,k})}} \int_{Q_{j,k}^{(a,b)}} ||\psi||^2 \frac{\rho(x^-,x'^-)^\beta (\delta r)^\gamma}{(\delta r + \rho(x^-, z_{j,k}))^{\mathrm{n}+\gamma+\beta}}$$
$$\cdot \frac{(\delta r)^\gamma}{(\delta r + \rho(y, z^-))^{\mathrm{n}+\gamma}} d\mu(y)$$
$$+ \sum_{\substack{k:\rho(x^-,x'^-)>\delta r+\rho(x^-,z_{j,k}) \text{ or} \\ \rho(x^-,x'^-)>\delta r+\rho(x'^-,z_{j,k})}} \int_{Q_{j,k}^{(a,b)}} ||\psi||^2 \frac{\rho(x^-, z_{j,k})^\beta (\delta r)^\gamma}{(\delta r + \rho(x^-, z_{j,k}))^{\mathrm{n}+\gamma+\beta}}$$
$$\cdot \frac{(\delta r)^\gamma}{(\delta r + \rho(y, z^-))^{\mathrm{n}+\gamma}} d\mu(y)$$
$$\leq C_{\beta,\gamma} ||\psi||^2 \int_0^{+\infty} \frac{\rho(x^-,x'^-)^\beta (\delta r)^\gamma}{(\delta r + \rho(x^-, z^-))^{\mathrm{n}+\gamma+\beta'} (\delta r + \rho(x^-, z^-))^{\beta-\beta'}} \frac{d\delta}{\delta}$$

(by Quasi-Orthogonality III)

$$+ 2||\psi||^2 \int_0^{+\infty} \frac{(1-b)^{-(\mathrm{n}+\gamma+\beta)} \rho(x^-,x'^-)^\beta (\delta r)^{\gamma-\beta}}{(\delta r + \rho(x^-, z^-))^{\mathrm{n}+\beta'} (\delta r + \rho(x^-, z^-))^{\gamma-\beta} (\delta r + \rho(x^-, z^-))^{\beta-\beta'}}$$

(since for k in the second sum we have $\rho(x^-, z_{j,k}) \leq \rho(x^-, x'^-)^\beta$, and by Quasi-Orthogonality III)

$$\leq 2^{\beta'-\beta}\left(C_{\beta,\gamma} + 2(1-b)^{-(\mathrm{n}+\gamma+\beta)}\right)||\psi||^2 \rho(x^-, x'^-)^{\beta'} \rho(x^-, z^-)^{-\mathrm{n}-\beta'}$$

(since $\rho(x^-, x'^-)^\beta \leq 2^{\beta'-\beta}\rho(x^-, x'^-)^{\beta'}(\delta r + \rho(x^-, z^-))^{\beta-\beta'}$ and by Lemma 5.6) .

We know pass to the smoothness estimate in z: for $\rho(z, z') \leq \frac{\rho(x,z)}{2}$ we have

$$|R_1(x, z) - R_1(x, z')| \leq$$

$$\leq \left|\sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \int_{a^{-j}}^{a^{-j+1}} \int_{Q_{j,k}^{(a,b)}} (T_x\psi_\delta(y) - T_x\psi_\delta(z_{j,k}))\,(T_y\psi_\delta(z) - T_y\psi_\delta(z'))\right|$$

$$\leq ||\psi||^2 \sum_{j\in\mathbb{Z}} \int_{a^{-j}}^{a^{-j+1}} \left(\sum_{\substack{k:\rho(z,z')\leq\delta r+\rho(z,y^-) \text{ or} \\ \rho(z,z')\leq\delta r+\rho(z',y^-)}} \int_{Q_{j,k}^{(a,b)}} \frac{(\delta r)^\gamma}{(\delta r + \rho(x^-, y))^{\mathrm{n}+\gamma}} \right.$$

$$\cdot \frac{\rho(z, z')^\beta (\delta r)^\gamma}{(\delta r + \rho(z, y^-))^{\mathrm{n}+\gamma}\,(\delta r)^\beta}$$

$$\left. + \sum_{\substack{k:\rho(z,z')>\delta r+\rho(z,y^-) \text{ or} \\ \rho(z,z')>\delta r+\rho(z',y^-)}} \int_{Q_{j,k}^{(a,b)}} \frac{(\delta r)^\gamma}{(\delta r + \rho(x^-, y))^{\mathrm{n}+\gamma}} \frac{\rho(y^-, z)^\beta (\delta r)^\gamma}{(\delta r + \rho(z, y^-))^{\mathrm{n}+\gamma}\,(\delta r)^\beta}\right) d\mu(y)\frac{d\delta}{\delta}$$

$$\leq 2(1-b)^{-(\mathrm{n}+\gamma+\beta)}||\psi||^2 \int_0^{+\infty} \int_X \frac{(\delta r)^\gamma}{(\delta r + \rho(x^-, y))^{\mathrm{n}+\gamma}} \frac{\rho(z, z')^\beta (\delta r)^\gamma}{(\delta r + \rho(z^-, y))^{\mathrm{n}+\gamma}\,(\delta r)^\beta} d\mu(y)\frac{d\delta}{\delta}$$

(since for k in the second sum we have $\rho(y^-, z) \leq \rho(z, z')$)

$$\leq ||\psi||^2 \int_0^{+\infty} \frac{2(1-b)^{-(\mathrm{n}+\gamma+\beta)}(\delta r)^\gamma(\delta r)^{-\beta}\rho(z, z')^\beta}{(\delta r + \rho(x^-, z^-))^{\mathrm{n}+\beta'}\,(\delta r + \rho(x^-, z^-))^{\gamma-\beta}\,(\delta r + \rho(x^-, z^-))^{\beta-\beta'}}\frac{d\delta}{\delta}$$

$$\leq ||\psi||^2 \int_0^{+\infty} \frac{2(1-b)^{-(\mathrm{n}+\gamma+\beta)}2^{\beta-\beta'}\rho(z, z')^{\beta'}}{(\delta r + \rho(x^-, z^-))^{\mathrm{n}+\beta'}\,(\delta r + \rho(x^-, z^-))^{\beta-\beta'}}\frac{d\delta}{\delta}$$

(since $\rho(z, z') \leq 2(\delta r + \rho(x^-, z^-))$)

$$\leq 2^{\mathrm{n}+\beta'-\beta}(1-b)^{-(\mathrm{n}+\gamma+\beta)}||\psi||^2\rho(z, z')^{\beta'}\rho(x^-, z^-)^{-\mathrm{n}-\beta'}.$$

We now show that R_1 satisfies (7.5). Suppose $a^{-j} \leq \delta \leq a^{-j+1}$, $f \in \mathcal{C}_0^\eta(X \times G)$ with supp.$f \subset Q = B(x_1, r_1) \times B(y_1, r_1)$, $x_1, y_1 \in X$, $||f||_\infty \leq 1$, $||f(\cdot, y)||_\eta \leq r_1^{-\eta}$, $||f(x, \cdot)||_\eta \leq r_1^{-\eta}$, for all $x, y \in X$. Then

$$\begin{aligned}
|< f(x,\cdot), \psi_{\delta,y}(\cdot) >| &= \left|\int_X f(x, z)\psi_{\delta,y}(z)d\mu_X(z)\right| \\
&= \left|\int_X \left\{f(x, z) - f(x, y^-)\right\}\psi_{\delta,y}(z)d\mu_X(z)\right| \\
&\leq ||f(x,\cdot)||_\eta \int_X \rho(z, y^-)^\eta\,|\psi_{\delta,y}(z)|\,d\mu_X(z) \\
&\leq C\delta^\eta||f(x,\cdot)||_\eta \\
&\leq Ca^{-j\eta}r_1^{-\eta}\ .
\end{aligned}$$

But we can also estimate the above by

$$\begin{aligned}|< f(x,\cdot),\psi_{\delta,y}(\cdot)>| &\le C||f||_\infty\left(\delta^{-1}r_1\right)^{\mathrm{n}}\\ &\le Ca^{\mathrm{n}j}r_1^{\mathrm{n}}\ .\end{aligned}$$

We thus obtain

$$\begin{aligned}&\left|< R_1,f>_{\mathbb{L}^2(X\times X)}\right|\\ &\le\int_Q\sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)}\int_{a^{-j}}^{a^{-j+1}}\int_{Q_{j,k}^{(a,b)}}|\psi_{\delta,y}(x)-\psi_{\delta,k}(x)|\,|< f(x,\cdot),\psi_{\delta,y}(\cdot)>|\,d\mu(y)\frac{d\delta}{\delta}d\mu(x)\\ &\le\int_Q\sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)}\int_{a^{-j}}^{a^{-j+1}}\int_{Q_{j,k}^{(a,b)}}\frac{b^\beta\delta^\gamma}{(\delta+\rho(x^-,y))^{\mathrm{n}+\gamma}}\left(a^{-j\eta}r_1^{-\eta}\wedge a^{\mathrm{n}j}r_1^{\mathrm{n}}\right)d\mu(y)\frac{d\delta}{\delta}d\mu(x)\\ &\le 2b^\beta\int_Q\sum_{j\in\mathbb{Z}}\int_{a^{-j}}^{a^{-j+1}}\left(a^{-j\eta}r_1^{-\eta}\wedge a^{\mathrm{n}j}r_1^{\mathrm{n}}\right)\frac{d\delta}{\delta}d\mu(x)\\ &\le 2b^\beta(\ln a)|Q|\sum_{j\in\mathbb{Z}}\left(a^{-j\eta}r_1^{-\eta}\wedge a^{\mathrm{n}j}r_1^{\mathrm{n}}\right)\\ &\le 2b^\beta\frac{\ln a}{a-1}r_1^{\mathrm{n}}\end{aligned}$$

(by the estimate below)

$$\le 2b^\beta r_1^{\mathrm{n}}\ ,$$

which shows that R_1 satisfies the strong weak boundedness property. The estimate we used above is the following. Choose $j_0\in\mathbb{Z}$ so that $a^{-(j_0+1)(\mathrm{n}+\eta)}\le r^{\mathrm{n}+\eta}<a^{-j_0(\mathrm{n}+\eta)}$; then

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\left(a^{-j\eta}r^{-\eta}\wedge a^{\mathrm{n}j}r^{\mathrm{n}}\right) &\le\sum_{j>j_0}a^{-j\eta}r^{-\eta}+\sum_{j\le j_0}a^{\mathrm{n}j}r^{\mathrm{n}}\\ &\le C\frac{a^{-(j_0+1)\eta}}{1-a^{-\eta}}r^{-\eta}+C\frac{a^{\mathrm{n}j_0}}{1-a^{-1}}r^{\mathrm{n}}\\ &\le C\left(\frac{1}{1-a^{-\eta}}+\frac{1}{1-a^{-1}}\right)\\ &\le C\frac{1}{a-1}\ .\end{aligned}$$

Finally, to show that R_1 satisfies the double-smoothness condition (2.6), one divides the spaces into the sets $\{z\in X:\rho(x^-,x^{-\prime})\le a^{-j}+\rho(x^-,z^-),\rho(y,y')\le a^{-j}+\rho(y,z^-)\}$, $\{\rho(x^-,x^{-\prime})\le a^{-j}+\rho(x^-,z^-),\rho(y,y')>a^{-j}+\rho(y,z^-)\}$ and $\{\rho(x,x')>a^{-j}+\rho(x^-,z^-),\rho(y,y')>a^{-j}+\rho(y,z^-)\}$, and applies the usual estimates on each of these sets. The details are left to the reader.

9.2. **Estimates for R_2.** The kernel of the operator $R_2^{(a,b)}$ is

$$R_2^{(a,b)}(x,z)=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathcal{K}(j)}\int_{a^{-j}}^{a^{-j+1}}\int_{Q_{j,k}^{(a,b)}}\{\psi_{\delta,y}(z)-\psi_{\delta,k}(z)\}\,T_x\psi_\delta(z_{j,k})d\mu(y)\frac{d\delta}{\delta}.$$

We can proceed exacly as for $R_1^{(a,b)}$ once we observe that we can find a C, uniform in b for all small b, so that for $y\in Q_{j,k}^{(a,b)}$ and $a^{-j}\le\delta<a^{-j+1}$ we have

$$|T_x\psi_\delta(z_{j,k})|\le C\frac{(\delta r)^\gamma}{(\delta r+\rho(x,y^-))^{\mathrm{n}+\gamma}}.$$

9.3. **Estimates for** R_3**.** The kernel of the operator $R_3^{(a,b)}$ is

$$R_3^{(a,b)}(x,z) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathcal{K}(j)} \int_{a^{-j}}^{a^{-j+1}} \int_{Q_{j,k}^{(a,b)}} \left\{ T_x\psi_\delta(z_{j,k}) - \psi_{j,k}^{(a,b)}(x) \right\} T_{z_{j,k}}\psi_\delta(z) dy \frac{d\delta}{\delta}$$

Before proceeding $R_3^{(a,b)}(x,z)$, let us observe the following:

$$\begin{aligned}
\left| T_x\psi_\delta(z_{j,k}) - \left(\mu(Q_{j,k}^{(a,b)}\right)^{-\frac{1}{2}} \psi_{j,k}^{(a,b)}(x) \right| &= \left| T_x D_\delta \psi(z_{j,k}) - T_{z_{j,k}} D_{a^{-j}} \psi(x) \right| \\
&= \left| T_x \left(D_\delta - D_{a^{-j}} \right) \psi(z_{j,k}) \right| \\
&\leq ||D_\delta - D_{a^{-j}}||_{\mathcal{M}(0,\delta r,\beta,\gamma)} ||\psi|| \frac{(\delta r)^\gamma}{(\delta r + \rho(x^-, z_{j,k}))^{\mathrm{n}+\gamma}} \\
&\leq C_{\mathrm{d},\beta,\gamma} \left| \frac{\delta}{a^{-j}} - 1 \right|^{\mathrm{d}} ||\psi|| \frac{(\delta r)^\gamma}{(\delta r + \rho(x^-, z_{j,k}))^{\mathrm{n}+\gamma}} \\
&\leq C_{\mathrm{d},\beta,\gamma} (a-1)^{\mathrm{d}} ||\psi|| \frac{(\delta r)^\gamma}{(\delta r + \rho(x^-, z_{j,k}))^{\mathrm{n}+\gamma}}.
\end{aligned}$$

We then proceed like we did for $R_2^{(a,b)}(x,z)$, replacing the bounds dependent on b by the correspondent bounds dependent on $(a-1)$.

9.4. **Estimates for** R_4**.** The estimates for $R_4^{(a,b)}(x,z)$ are similar to the ones already carried out for $R_1^{(a,b)}(x,z)$ and $R_3^{(a,b)}(x,z)$.

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