Test 1 - Math 133
Tuesday, Feb. 16th

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Office hours: Tuesday 4pm-5:30pm.

www.math.duke.edu/~ mauro/teaching.html

Write your name at the top right corner of each page. Provide full answers and motivations. No credit is given for non-motivated answers. No books, notes, or calculators are allowed. You have 1 hour and 10 minutes.

1. (a) When do we say that a problem is well posed? [5 pts] Is $u_t - u_x = 0$, $x \in \mathbb{R}$, $t > 0$ well-posed? [3 pts]

A problem is well-posed if it has a unique solution that depends continuously on the initial data.
$u_t - u_x = 0$ is not well-posed since it has infinitely many solutions (of the form $f(x+t)$, $f \in C^2$).

(b) Is $u + (1-u)u_x + u_t = 0$ a first order PDE? [3 pts] Is it linear? [3 pts]

It is first order since only derivatives of order 1 appear.
It is not linear because of the nonlinear term uu_x: if u_1, u_2 are solutions, in general $u_1 + u_2$ will not be.

(c) For the following PDE's, indicate whether the characteristics are lines or not, and whether the solutions are constant along the characteristics or not [4 pts each]:

i. $u_t + cu_x = 0$

They are straight lines since c is the constant propagation speed; the sol. $f(x-ct)$ is constant along each characteristic.

ii. $u_t + c(x)u_x = 0$

They are not straight lines since c depends in general on x (spatial variable). The solution is constant along each characteristic since the eqn. is homogeneous, so that $\langle Du, (1, c(x)) \rangle = 0$.

iii. $u_t + (F(u))_x = 0$

The characteristics are lines but they may intersect. If they do not intersect then the solution is constant on each characteristic. When they intersect, the behaviour may be more complicated and depends on F and

iv. $u_t + uu_x = 0$

Particular case of (iii) with $F(u) = \frac{u^2}{2}$.

v. $u_t + 3u_x = x$

They are lines (as in (i)), but u is not constant along them since not homogeneous eqn.

vi. $u_t + xu_x = 0$

Not lines since c is spatially dependent (as in (ii)). Sol's constant on characteristics since the eqn. is homogeneous.

(d) Solve either one of the last two PDE's above, with initial condition $u(x,0) = f(x)$ [30 pts].

v. $u_t + 3u_x = x$: Let $v(t) = u(x_0 + 3t, t)$, so that

$\frac{dv}{dt}(t) = (u_x + 3u_t)(x_0 + 3t, t) \underset{\text{by PDE}}{=} x_0 + 3t$

$v(t) = x_0 t + \frac{3}{2}t^2 + c \Rightarrow v(0) = c = u(x_0, 0) = f(x_0)$

$\Rightarrow u(x_0 + 3t, t) = x_0 t + \frac{3}{2}t^2 + f(x_0)$

$\Rightarrow u(x,t) = (x - 3t)t + \frac{3}{2}t^2 + f(x - 3t)$

VI. $u_t + xu_x = 0$: $\left\langle \begin{pmatrix} u_t \\ u_x \end{pmatrix}, \begin{pmatrix} 1 \\ x \end{pmatrix} \right\rangle = 0 \Rightarrow$ eqn for characteristics parametrized by $t \mapsto x(t)$ is

$\dot{x}(t) = x(t) \Rightarrow x(t) = ce^t$

If $x_0 = x(0)$, then $x(t) = x_0 e^t \Rightarrow x_0 = x(t)e^{-t}$

$\Rightarrow u(x,t) = f(xe^{-t})$ is the solution.

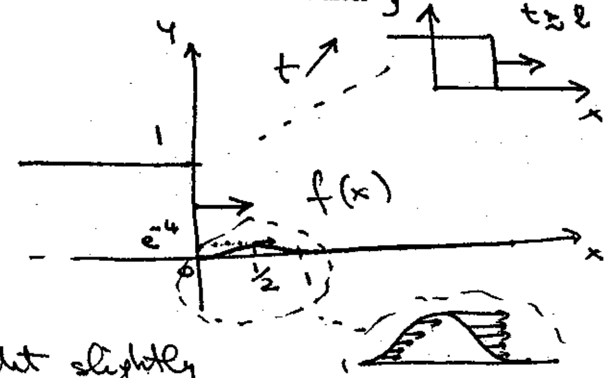
2. Consider the equation $u_t + uu_x = 0$, $x \in \mathbb{R}$, $t > 0$, with initial condition $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} 1 & ,x \leq 0 \\ \exp\left\{-\frac{1}{x(1-x)}\right\} & ,0 \leq x \leq 1 \\ 0 & ,x \geq 1 \end{cases}$$

Which type of PDE is this? [5 pts] Is this problem well-posed? [3 pts]

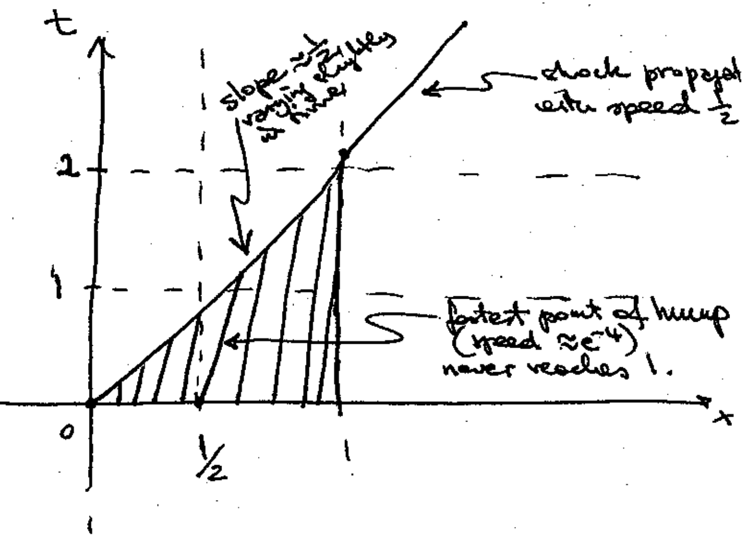
First order nonlinear PDE, from conservation law with flux $F(u) = \frac{u^2}{2}$. f is discontinuous so we seek weak sol.'s which are stable. The pb. per se is not well-posed since multiple sol.'s may exist.

Describe the solution, in words and pictorially, through the characteristics and their behavior, for $t > 0$ [30 pts]. Do shocks develop? Why? [10 pts] If they do develop, describe the solution after the shock forms [15 pts].



The initial condition has a discontinuity at 0, so that a shock will propagate from left to right (since $\lim_{x\to 0^-} f(x) > \lim_{x\to 0^+} f(x)$) with speed initially equal to $\frac{\lim_{x\to 0^-} f(x) - \lim_{x\to 0^+} f(x)}{2} = \frac{1}{2}$.

As the shock advances, its height slightly decreases since it absorbs slowly moving parts of the profile of f. However since $\max_{x\in(0,1)} f(x) = f(\frac{1}{2}) = e^{-4}$, such decrease in speed is very small. The decreasing part of the hump of f moves towards 1 with maximum speed $e^{-4} \ll \frac{1}{2}$, and it could potentially generate a second shock. This however does not happen, as the characteristics show, since the shock starting at 0 moves much faster and "absorbs" the hump before the second shock may be generated. Once the shock started at 0 reaches 1, it will propagate with speed $\frac{1}{2}$ to the right.



3. Consider the heat equation $u_t = \Delta_x u$, for $x \in \mathbb{R}$, $t > 0$, and initial condition $u(x,0) = f(x)$, where $f \in C^2(\mathbb{R})$, $0 \le f(x) \le 1$, and $f(x) \to 0$ as $|x| \to +\infty$.

(a) State a maximum principle for u [15 pts].

Since f is bounded on C^2, the solution $u(x,t)$ is bounded above by 1 and has no local maxima in $\mathbb{R} \times (0, +\infty)$.

By considering $-u$, in fact one can show that u is also ≥ 0.

(b) Write the solution $u(x,t)$ of the given IVP, in terms of the fundamental solution [8 pts]. Then prove or disprove: $u(x,t) \to 0$ uniformly on $[0,1]$ (*).

Let $G(x,t)$ be the fundamental solution $G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$. Then

$u(x,t) = \int_{\mathbb{R}} G(x-y,t) f(y)\,dy.$

For $x \in [0,1]$,

$$|u(x,t)| = \left| \int_{\mathbb{R}} G(x-y,t) f(y)\,dy \right| \le \int_{-M}^{M} \left| \frac{1}{\sqrt{4\pi t}} \underbrace{e^{-\frac{(x-y)^2}{4t}}}_{\le 1} \underbrace{f(y)}_{\le 1} \right| dy + \longrightarrow$$

other side

(c) Suppose that, with f as above, we know that $f(x) = 0$ for $x \in [a,b]$. Is that true that $u(x,t) = 0$ for $x \in [a,b]$, for all $t > 0$? For at least some $t > 0$? Why? What if a and/or b are infinity? [10 pts]

No (unless $f \equiv 0$ everywhere): from the formula in (b) we see that, since $G > 0$ for all $x \in \mathbb{R}$, $t > 0$, $u > 0$ everywhere (since $f \ge 0$) unless $f \equiv 0$ on $\mathbb{R}$.

For no $t > 0$ is $u(x,t)$ identically zero on $[a,b]$, for the same reason.

The same reasoning applies to the case when $a, b = \pm\infty$.

This is the infinite speed of propagation of the heat kernel.

$$\longrightarrow + \left| \left(\int_{-\infty}^{-M} + \int_{M}^{+\infty} \right) G(x-y,t) \underbrace{\cancel{f(y)}}_{\leq 1} dy \right|$$

$$\underset{\substack{\text{use} \\ x\in[0,1]}}{\leq} \underbrace{\frac{2M}{\sqrt{4\pi t}} + \int_{-\infty}^{-M} \overbrace{G(0-y,t)}^{\leq \frac{1}{\sqrt{4\pi t}} e^{-\frac{M^2}{4t}}} + \int_{M}^{+\infty} \overbrace{G(1-y,t)}^{\leq \frac{1}{\sqrt{4\pi t}} e^{-\frac{(1-M)^2}{4t}}}}_{\text{bound independent of } x} \xrightarrow[t \to +\infty]{} 0$$

(d) Now consider the same heat equation on $[0,1]$, with Neumann boundary conditions.

i. Write explicitly the boundary conditions [5 pts].

$\partial_x u(x,t) = 0$ for all t and $x = 0, 1$

ii. Use separation of variables to find first particular solutions of the form $\varphi_n(x)\psi_n(t)$, and the eigenvalues λ_n of the problem [30 pts].

From the PDE we obtain

$$\varphi_n(x)\psi_n'(t) = \varphi_n''(x)\psi_n(t)$$

$$\Rightarrow \frac{\psi_n'(t)}{\psi_n(t)} = \frac{\varphi_n''(x)}{\varphi_n(x)} \Rightarrow \begin{cases} \psi_n'(t) = -\lambda\psi_n(t) \\ \varphi_n''(x) = -\lambda\varphi_n(x) \end{cases}$$

$\Rightarrow \varphi_n(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$ (since for $\lambda < 0$ there are no sol's satisfying the ∂ cond's)

$\Rightarrow 0 = \partial_x\varphi_n(0) = \sqrt{\lambda}A\cos(\sqrt{\lambda}\cancel{0}) + \cancel{B\sqrt{\lambda}} = A \Rightarrow A = 0$ or $\lambda = 0$ (const. sol.)

$0 = \partial_x\varphi_n(1) = \cancel{\sqrt{\lambda}}B\sin(\sqrt{\lambda}\cdot 1)$

$\Rightarrow \sqrt{\lambda} = n\pi \Rightarrow \boxed{\lambda_n = (n\pi)^2}$, $n = 1, 2, \ldots$ or $n = 0$

$\Rightarrow \psi_n'(t) = -(n\pi)^2\psi_n(t)$

$\Rightarrow \psi_n(t) = e^{-(n\pi)^2 t}\psi_n(0)$

$\varphi_n(x)\psi_n(t) = \cos(n\pi x)e^{-(n\pi)^2 t}$, $n = 0, 1, 2 \ldots$

iii. Assume the initial condition is $f(x) = \cos(4\pi x)$. Write the solution of the heat equation with the Neumann boundary conditions and this initial condition (*). What can you say about the $L^2([0,1])$ norm of the solution $u(\cdot, t)$, for any fixed t (**)?

By the general theory, $\left\{ \frac{\varphi_n}{\|\varphi_n\|_2} \right\}$ is o.n. basis in $L^2([0,1])$

Observe that $f = \varphi_4$, so that

$< f, \varphi_i > = 0$ unless $i = 4$ (by orthogonality)

Therefore

$$u(x,t) = \sum_{n=0}^{+\infty} \overbrace{\cancel{< f, \frac{\varphi_n}{\|\varphi_n\|_2} >}}^{0 \text{ unless } n=4} \frac{\varphi_n}{\|\varphi_n\|_2}(x)\, \psi_n(t)$$

$$= < f, \varphi_4 / \|\varphi_4\|_2 > \frac{\varphi_4}{\|\varphi_4\|_2}(x)\, \psi_4(t)$$

$$= \frac{1}{\|\cos(4\pi x)\|^2_{L^2([0,1])}} \cos(4\pi x)\, e^{-(4\pi)^2 t}$$

$$\| u(\cdot, t) \|_{L^2} = \left\| \frac{\cos(4\pi x)}{\|\cos(4\pi x)\|^2} \right\| e^{-(4\pi)^2 t} =$$

$$= \frac{1}{\|\cos(4\pi x)\|^2} e^{-(4\pi)^2 t}$$

$\Rightarrow$ the L^2 norm of $u(\cdot, t)$ decays exponentially fast as $t \to +\infty$.