Homework 1 - Math 225
Due Thursday, Jan. 22

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Office hours: by appointment.

www.math.duke.edu/~mauro/teaching.html

I prefer homework written in pen rather than pencil. The handwriting and organization of your work on the page should be clear. Include appropriate explanations for what you are doing in your calculations and why, and what conclusions you draw or observations you make.

The homework should include a printout of the Matlab/C/Fortran code you used and of the code output (including figures as needed/requested). Also send me a copy of the code via e-mail: if you have multiple files, compress them into a unique zip file. Name the file as FamilyName_FirstInitial_Homework_xx.zip, where xx is the homework number. This will apply to all the future homework as well.

1. Write a program to sum N numbers randomly distributed in $[0, 1]$, in different ways:
   - one after the other as they are drawn;
   - sorted in decreasing order (requires storing the N numbers);
   - sorted in increasing order (requires storing the N numbers);
   - using the following algorithm:
     ```
     sum =0.0;
     c = 0.0;
     for k = 2:n
     y = a(k) - c;
     t = sum + y;
     c = (t - sum) - y;
     sum = t;
     end;
     ```

   Which algorithm is more accurate? Why?

   What would you do if the numbers were drawn from the uniform distribution on $[-1, 1]$?

2. Let
$$l_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$
be the Legendre interpolating polynomial of degree n corresponding to $n + 1$ distinct points $x_0, \ldots, x_n$. Show that
$$\sum_{i=0}^{n} l_i(x) = 1$$
for all x.

3. Let $x_0, \ldots, x_n$ be distinct points in $\mathbb{R}$, and consider the problem of interpolating a function at these points with a function in the form
$$P_n(x) = \sum_{j=0}^{n} c_j e^{jx} ,$$
i.e. we are looking at finding P_n in this form so that $P_n(x_i) = y_i$, for $i = 0, \ldots, n$. Show that such P_n exists and is unique.

4. Let $f(x) = e^x$. Let $p(x)$ be the interpolating polynomial of degree one passing through the points $(0,\,1)$ and $(2,\,e^2)$.

   (a) For $x = 3/2$, find the values (or value) of ξ that make(s) the following error formula exact:

$$f(x) - \sum_{j=0}^{n} f(x_j) l_j(t) = \frac{(x - x_0) \cdot \cdots \cdot (x - x_n)}{(n+1)!} f^{(n+1)}(\xi)\,.$$

   Do the same for $x = 0$, $x = 1/2$, and $x = 1$.

   (b) Let $E(x) = f(x) - p(x)$. Use the error formula to obtain a polynomial $B(x)$ that is a good approximation to $E(x)$ on the interval [0,2] (this polynomial is not unique).

   (c) Graph $B(x)$ and $E(x)$; also graph $f(x)$, $p(x)$, and $p(x) + B(x)$.

5. (a) Write a program that computes Newton divided differences. Attach a hard copy of your program.

   (b) Use the program to obtain explicit representations for the interpolating polynomials of degree 2, 6, and 10 to $f(x) = 1/(1+x^2)$, $x \in [-5, 5]$, with evenly spaced points (enough of them to obtain good resolution graphs). Discuss.