Homework 9- Math 225
Due Tuesday, Apr. 21th

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Office hours: by appointment.

www.math.duke.edu/~ mauro/teaching.html

I prefer homework written in pen rather than pencil. The handwriting and organization of your work on the page should be clear. Include appropriate explanations for what you are doing in your calculations and why, and what conclusions you draw or observations you make.

The homework should include a printout of the Matlab/C/Fortran code you used and of the code output (including figures as needed/requested). Also send me a copy of the code via e-mail: if you have multiple files, compress them into a unique zip file. Name the file as FamilyName_FirstInitial_Homework_xx.zip, where xx is the homework number. This will apply to all the future homework as well. Please use the subject "Math 225 homework" in your e-mail.

1. Simulate Brownian motion on $[0, 1]$ in each of the 3 ways discussed in class: by random walks with Gaussian increments, by rescaled random walks with Bernoulli increments, by Fourier expansions.

   Discuss the properties of these paths as the discretization step Δt gets smaller.

   What can you say about the empirical moments $\mathbb{E}[W_t]$ and $\mathrm{Var}[W_t]$ of the generated processes? What about $\mathrm{cov}(W_s, W_t)$, for various values of $t - s$ (if $t > s$)?

   What is the computational complexity of the methods as a function of the number of discretization points?

2. Consider the Brownian bridge process on $[0, 1]$ from 0 to y, defined by

   $$W^{(1,y)}_{(0,0)}(t, \omega) = W(t, \omega) - t(W(1, \omega) - y)$$

   whose paths start at 0 and (a.s.) end (at time 1) at y:

   Compute $\mathbb{E}[W^{(1,y)}_{(0,0)}(t, \cdot)]$, $\mathrm{Var}[W^{(1,y)}_{(0,0)}(t, \cdot)]$ and $\mathrm{cov}(W^{(1,y)}_{(0,0)}(s, \cdot), W^{(1,y)}_{(0,0)}(t, \cdot))$.

   How do you generate Brownian bridge sample paths?

   Answer the same questions as above for the Brownian bridge process. You may skip the computation of the Fourier expansion if you are really uncomfortable with Fourier analysis, but it is not hard at all, and you'll get extra credit for it.

3. Simulate the solution of $dX_t = a\, X_t\, dt + b\, X_t\, dW_t$ (with a, b constants) by linearly interpolating an Euler approximation $Y_{n+1} = Y_n + aY_n\Delta t + bY_n\Delta W_n$, where Δt is the (uniform) stepsize on $[0, 1]$ (and ΔW is the corresponding Brownian step, that you can simulate with an appropriately rescaled normal draw), and Y_n is the estimated solution at the point $n\Delta t$, for $n = 0, \dots, 1/\Delta t$. You may let the initial condition be $Y_0 = 1$. Compare with the true solution, for different values of the stepsize Δt, and discuss your results.