Homework 1 - Due Fri. Feb. 3rd
AS.110.446, EN.550.416

Dr. Mauro Maggioni **Office**: 405 Krieger Hall
Web page: www.math.jhu.edu/~ mauro **E-mail**: myfirstname.maggioni@jhu.edu

Homework Policies

Homework is due at the beginning of class, stapled, written legibly, on one side of each page only. Otherwise, it will be returned ungraded.
Some problems from the homework will reappear on exams. The lowest homework score will be dropped. No late homework will be accepted. Johns Hopkins' policies apply with no exceptions to cases of incapacitating short-term illness, or for officially recognized religious holiday. You may, and are encouraged to, discuss issues raised by the class or the homework problems with your fellow students and both offer and receive advice. However all submitted homework must be written up individually without consulting anyone else's written solution.

The submission of homework that require numerical work on a computer should include the following: printout of the code used to solve the problems, of its inputs and of its outputs. The code should be written clearly, copiously commented, and input/outputs of the code clearly documented in format and content. The specific outputs requested by the exercise should be discussed in your writeup as needed in order to answer the questions in the problems. For example if the problem asks you to compare the results of two algorithms for solving a given linear system, you should exhibit the the code for the two algorithms, commented, the input to the algorithms and the two outputs, and comment on whether the results are the same or not, why, etc...

We can help with Matlab, R, C/C++, less with other languages, especially if the issues you encounter are language-specific rather than algorithm-related. The code provided with the course will be in Matlab, and so will be the links to toolboxes that may be required. Let us know if you have no access to computers with any of the above languages/compilers/software packages installed.

Feb. 1st: Typos corrected in red, and also added a few hints since in class I did not get to a couple of things that I wanted to say before you worked on this homework set.

Assignment

Review your linear algebra! Review carefully any concepts you are not completely familiar with, by going back to your linear algebra textbook as needed. On the wiki you may find some references. A good textbook for basic linear algebra is Halmos' book "Finite dimensional vector spaces". The connection between matrices, linear operators, bases, changes of bases, and the geometric interpretation of all of the above, are sometimes not emphasized enough in linear algebra courses, but are fundamental for this course. Go through examples and short exercises proposed in class and do them.

Exercises

Exercise 1 (25pts). Pick two distinct bases of $\mathbb{R}^2$, and a linear operator $A : \mathbb{R}^2 \to \mathbb{R}^2$, different from the identity and surjective (onto).

- Represent this operator with respect the two bases, obtaining two matrices $[A]_1$ and $[A]_2$.
- Is the rank of the two matrices is the same? Why/why not?
- Find a unit vector x such that $||Ax||$ is largest possible. In order to compute this you may use one the representations of A that you calculated, $[A]_1$ or $[A]_2$. Does the result depend on which representation you use for your calculations?

Exercise 2 (25pts). A linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is called *symmetric* if $\langle Av, w\rangle = \langle v, Aw\rangle$ for every $v, w \in \mathbb{R}^n$.

- Show that representing a symmetric linear operator A with respect to the standard basis yields a symmetric matrix.
- Show that representing a symmetric linear operator A with respect to any orthonormal basis yields a symmetric matrix.
- Show that representing a symmetric linear operator A with respect to a non-orthonormal basis may yield a non-symmetric matrix. [Note that in order to prove this, you only need to exhibit one example when this happens, for example in $\mathbb{R}^2$.]

Exercise 3 (25pts*). [Vandermonde Matrix] Fix $\{x_i\}_{i=1}^m$. Consider the map $\mathbb{R}^{n+1} \to \mathbb{R}^m$ defined by $c = (c_0, \dots, c_n) \mapsto (p_c(x_1), \dots, p_c(x_m))$, where $p_c = \sum_{l=0}^n c_l x^l$, i.e. from vectors of coefficients of polynomials of degree $< n$ to vectors $(p(x_i))_{i=1}^m$.

- Show that this map is linear
- Show that this map is represented, upon choosing the standard basis in $\mathbb{R}^{n+1}$ and $\mathbb{R}^m$, by the $m \times n$ Vendermonde matrix

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{bmatrix}.$$

  Therefore if $c = [c_0; c_1; \dots; c_n]$ is the column vector of coefficients of p, i.e. $p(x) = c_0 + c_1 x + c_2 x^2 \cdots + c_{n-1} x^n$, then Ac gives the the sampled polynomial values, i.e. $(Ac)_i = p(x_i)$, for $i = 1, \dots, m$.

- When is this linear map invertible?

Exercise 4 (25pts). Prove that if O is an orthogonal $n \times n$ matrix, i.e. such that $O^T O = OO^T = I$, representing a linear map $\mathbb{R}^n \to \mathbb{R}^n$, and $|| \cdot ||$ is the Euclidean norm, then

- $||Ov|| = ||v||$ and $||Ov - Ow|| = ||v - w||$ for all $v, w \in \mathbb{R}^n$
- $\langle Ov, Ow\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^n$. Hint: use $\langle v, w\rangle = v^T w$, and also $||v||^2 = \langle v, v\rangle = v^T v$.

Then show if A is a linear map $\mathbb{R}^n \to \mathbb{R}^n$ then the following are equivalent (i.e. each one implies the other):

- $||Av|| = ||v||$ for every $v \in \mathbb{R}^n$;
- $\langle Av, Aw\rangle = \langle v, w\rangle$ for every $v, w \in \mathbb{R}^n$.