Homework 5 - Due Mon. Mar. 6th
AS.110.446, EN.550.416

Dr. Mauro Maggioni | **Office**: 405 Krieger Hall
Web page: www.math.jhu.edu/~ mauro | **E-mail**: myfirstname.maggioni@jhu.edu

Homework Policies

As in the first homework set.

Exercises

Exercise 1 (100 pts). Consider the space $L^2([-\frac{1}{2},\frac{1}{2}]) = \{g : \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(x)|^2 dx < +\infty\}$ with the norm $||g|| := \sqrt{\int_{-\frac{1}{2}}^{\frac{1}{2}} |g(x)|^2 dx}$, associated to the inner product $\langle g, h\rangle := \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x)\overline{h(x)}dx$.

- Show that $\{\mathbf{1}, \sqrt{2}\cos(2\pi nx), \sqrt{2}\sin(2\pi mx)\}_{m,n\geq 1}$ is an orthonormal set (here $\mathbf{1}$ denotes the constant function 1). To simplify the notation in what follows let
$$(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \dots) := \{\mathbf{1}, \sqrt{2}\cos(2\pi x), \sqrt{2}\sin(2\pi x), \sqrt{2}\cos(2\pi 2x), \sqrt{2}\sin(2\pi 2x), \dots\}\,.$$
This is in fact an orthonormal basis: recall that this means that for any $f \in L^2([-\frac{1}{2},\frac{1}{2}])$ we have
$$\left\| f - \sum_{l=1}^{L} \langle f, \varphi_l\rangle \varphi_l \right\|_{\mathbb{L}^2} \to 0 \qquad \text{as} \quad L \to +\infty\,,$$
and prove this statement by using what we stated in class, i.e. that $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$ is an orthonormal basis.

- We let $P_L f = \sum_{l=1}^{L} < f, \varphi_l > \varphi_l$ be the L-th partial sum used above. Interpret this as an orthogonal projection of f onto the span of the first L Fourier modes.

- Sample the interval $[-\frac{1}{2},\frac{1}{2}]$ at p equispaced points $x_1, \dots, x_p$ (for example using the `linspace` function in Matlab), and implement a function that computes $P_L f$ for any given f (sampled at the points x_i) and given $L > 0$. Check your function is reasonable by using it to (approximately) verify that $\{\varphi_l\}_{l=1}^{L}$, for moderate values of L, is an orthonormal basis. Discuss how this may be implemented in matrix form, and how many basic computations this function takes, as a function of p and L.

- Choose various functions (for example $f(x) = e^{\sin(2\pi x)}$ or $f(x) = \mathbf{1}_{[-\frac{1}{2},0]}(x) + \mathbf{1}_{[0,\frac{1}{2})}(x)$ and/or others) and a suitable p (e.g. $p = 1024$, or multiples thereof), and study $||f - P_L f||$ as a function of L: this is monotonically decreasing (why?) and tending to 0 as $L \to +\infty$, and if plotted in $\log_{10}$ scale you may see a trend in the decay rate. Try this for smooth periodic functions (recall: a function is periodic (with period 1) if $f(x+1) = f(x)$ for all x, i.e. f "repeats itself" from each interval I of width 1 to any interval $I + k$, for any k positive or negative integer), and for periodic functions that are not smooth (e.g. they have one or more jump discontinuities).

- The same as the previous point, but by adding Gaussian noise of size σ (e.g. for $\sigma = 0.01, 0.1$) to each of the functions f you considered, obtaining a sampled function $\tilde{f}$ [1].
  - Let $f = 0$, i.e. the constant function equal to 0 everywhere. What does $\tilde{f}$ look like? And $\widehat{\tilde{f}}$? How does $\widehat{\tilde{f}}$ depend on the noise level σ?

[1] you may do this in Matlab as follows: if `f` is a vector consisting of the values of the function f at various locations, you may let `f_tilde=f+`σ`randn(size(f));`.

- Look at $P_L\tilde{f}$ for a smooth (nontrival/nonconstant) f: How do the approximations $P_L f$ look like compared to the noisy f? More or less noisy? Explain why $P_L\tilde{f}$ sometimes looks like a "de-noised" version of f. Then comment on what happens when L becomes too large.
- Can you tell what $||\tilde{f} - P_L\tilde{f}||$ tends to as $L \to +\infty$?
- If you look instead at $||f - P_L\tilde{f}||$, how does this "error" behave as a function of L? Is it still monotone? Make sure you try both a smooth 1-periodic function such as $e^{\sin(2\pi x)}$ and a non-smooth 1-periodic function such as (the periodization of) $f(x) = \mathbf{1}_{[-\frac{1}{2},0]}(x) + \mathbf{1}_{[0,\frac{1}{2})}(x)$, as the answer to the last question may depend on the smoothness of f, at least in the regimes of values of p, L and σ that you are exploring. In this example it may pay off to choose σ "rather large".

These experiments needs to be done a bit carefully: make sure you keep L much smaller than the number of sample points p, e.g. $L \ll \sqrt{p}$ should suffice.