Homework 6 - Due Mon. Mar. 13th
AS.110.446, EN.550.416

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Homework Policies

As in the first homework set.

Exercises

Exercise 1 (50 pts). Consider a function which is C-Lipschitz, i.e. for all $x, y \in \mathbb{R}$ it holds that $|f(x) - f(y)| \leq C|x - y|$. What can you say about $|\langle f, \varphi_{j,k}\rangle|$, where φ is the Haar scaling function? What about $|\langle f, \psi_{j,k}\rangle|$, where ψ is the Haar wavelet?

Hint: at some point it may be useful to notice that $f(x) = f(x) - f(y) + f(y)$.

Exercise 2 (50 pts). Construct the function $f(x) = \sqrt{x} + \mathbf{1}_{x>0}(x) \cdot \sin\left(\frac{x}{\frac{1}{2}+\frac{1}{100}-x}\right)$ (where $\mathbf{1}_{x>0}(x) = 1$ if $x > 0$ and 0 otherwise) on $[-1/2, 1/2]$, and sample it at 16384 equi-spaced points.

Compute the Fourier transform and comment on its properties (e.g. decay).

Using `wavemenu` in `Matlab` analyze this function using Haar wavelets and Daubechies-4 wavelets. Describe the two dimensional digram of wavelet coefficients as a function of scale and location, describe how the wavelet transforms captures the discontinuity of the derivative at 0 and the oscillations near $\frac{1}{2}$. Make sure you compute as many scales as possible (10 or more if you can). Attach a couple of illustrative pictures of the two dimensional wavelet coefficient diagrams. Finally, go into the "compression", select the threshold method "remove near 0", and for various settings of the global threshold, look and comment on the approximations you obtain. Do this with both Haar and Daubechies wavelets. Describe what you see and why, and attach a couple of illustrative pictures.

Review questions in view of the midterm exam: these are not graded, and are for you to review before the midterm next week.

- Linear Algebra
  - Consider two bases B in vector space V of dimension m. Let A be a linear operator $V \to V$. What does it mean to represent A with respect to the basis B?
  - Define what an inner product on a vector space is, and what a norm induced by an inner product is.
  - Define what a basis for a vector space is.
  - Consider the vector space $\mathcal{C}([0, 1])$ of continuous functions on $[0, 1]$. Which ones of the following are subspaces (and why?):
    - $\{f : f(0) = 0\}$
    - $\{f : f(1) \leq 1\}$
    - $\{f : \int_0^1 f = 0\}$
    - $\{f : \int_0^1 f = 1\}$
  - Consider the space $\mathcal{P}^2([0, 1])$ of polynomials of degree up to 2 on the interval $[0, 1]$, with the usual L^2 inner product. Consider the subspace $S := \langle 1, x\rangle$. Project $x^2 + 1$ orthogonally onto this subspace. Then find the polynomial $p \in S$ such that $||p - (x^2 + 1)||_2$ is minimal.
- SVD.

- Define what the SVD of a matrix $X \in \mathbb{R}^{D \times n}$.
- Describe how to it can be used to find a k-dimensional affine subspace S_k such that the points $x_i \in \mathbb{R}^D$ defined by the columns of X are well-approximated by their orthogonal projection onto S_k.
- Which notion of distance between the points x_i and the plane S_k is minimized by S_k, over all sets of possible k-dimensional affine subspaces?
- If $A \in \mathbb{R}^{m \times n}$, with $m > n$, what does it mean to solve a linear system $Ax = b$, with A, b given, in the least squares sense?
- How can the SVD of a full rank matrix $A \in \mathbb{R}^{m \times n}$, with $m > n$, be used to solve $Ax = b$ in the least squares sense?
- Give an example, in detail, of a least squares problem that could arise when trying to fit an unknown function f on $[0, 1]$ given a set of pairs $(x_i, f(x_i))$.

• Fourier Analysis

- Consider $L^2([-\frac{1}{2}, \frac{1}{2}])$. Show that $E := \{f \in L^2([-\frac{1}{2}, \frac{1}{2}]) : f(-x) = f(x)\}$ is a subspace of L^2, consisting of even functions. Find an orthonormal basis for E.
- Compute the Fourier coefficients of the function $f(x) = x$ in $L^2([-\frac{1}{2}, \frac{1}{2}])$. Then apply Parseval's formula relating $||f||_2^2$ to the ℓ^2 norm of the Fourier coefficients to find a fun formula (I believe originally discovered by Euler).