Homework 2 - Due Wed. Sep. 9th
Math 561

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Homework Policies

As in homework 1.

Exercises

Exercise 1 (10 pts). Let A be an invertible $n \times n$ matrix. Show that:

- The columns of A are a basis for $\mathbb{R}^n$.
- The rows of A are a basis for $\mathbb{R}^n$.
- $Ax = 0$ implies $x = 0$, i.e. $\ker(A) = \{0\}$, where ker (sometimes also denoted by null(A)) is the kernel (or null space) of A.
- $Ax = b$ has a unique solution $x \in \mathbb{R}^n$ for any choice of $b \in \mathbb{R}^n$.
- $\text{range}(A) = \mathbb{R}^n$.
- If $v \in \mathbb{R}^n$ is written in the standard basis of $\mathbb{R}^n$, then its coordinates in the basis of columns of A is given by $A^{-1}v$.

Exercise 2 (10 pts). We want to start understanding the computational cost of two matrix operations by using Matlab (you may use the `cputime` function to measure time).

- For a $m \times m$ matrix A and a fixed vector $v \in \mathbb{R}^n$, compute the time it takes to compute Av in Matlab, and plot such time for $m = 2^j$, $j = 1, 2, \ldots, 12$. Does the computation take a time proportional to which power of n? Can you explain why?
- For a $m \times n$ matrix A, compute the time it takes to compute AA^T in Matlab, and plot such time for $n, m = 2^j$, $j = 1, 2, \ldots, 12$, and study the computation time as a function of m, n. Does the computation take a time proportional to which power of n, m? Can you explain why? You may push j to higher values to see these trends better, but watch out for memory usage and speed of computation.

Technical note: many computers now have multiple cores and/or multiple CPU's. `cputime` measures the total cpu time spent computing, even if the computation was run in parallel, at least to a certain degree, across multiple cores/CPU's. The actual real world time spent computing may scale differently, and it is measured by the functions `tic` and `toc`. You may check how many cores/CPU's your computer has, how many are used during the computations above, and if the time measured by `cputime` coincides or not with that measured by `tic` and `roc`.

Exercise 3 (From Trefethen's book, 10 pts). Show that for $u, v \in \mathbb{R}^m$, if $A = I + uv^T$ is invertible, then its inverse has the form $A^{-1} = I + \alpha uv^T$, and give an expression for α. For what u, v is A singular? When A is singular, what is $\ker(A)$, the kernel (or null space) of A?

Exercise 4 (From Trefethen's book, 10 pts). Show that for $u, v \in \mathbb{R}^m$, show that $||uv^T||_2 = ||u||_2||v||_2$. Does the same hold for the Frobenius norm?

Exercise 5 (From Trefethen's book, 15 pts). Consider the vector space $\mathbb{R}^n$ with the norms:

$$||v||_p := \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}},$$

for $p \geq 1$, and

$$||v||_\infty := \max_{i=1,\ldots,n} |v_i| \,.$$

These are all norms (you may check easily that they are, except for the triangle inequality, which is not completely trivial to show, except in the case $p = 2$, i.e. for the Euclidean norm, and for the case $p = +\infty$).

A norm on $\mathbb{R}^n$ induces a norm on linear operators $\mathbb{R}^n \to \mathbb{R}^m$ via

$$||A||_p := ||A||_{p\to p} := \max_{||v||_p=1} ||Av||_p \,.$$

Study these materials in Trefethen's book. Then prove the following inequalities, and show an example of a nonzero vector or matrix, for general m, n, for which equality is achieved: for $v \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$:

- $||x||_\infty \leq ||x||_2$
- $||x||_2 \leq \sqrt{m}||x||_\infty$
- $||A||_\infty \leq \sqrt{n}||A||_2$
- $||A||_2 \leq \sqrt{m}||A||_\infty$