Review problems
Introduction to Harmonic Analysis and its Applications

Dr. Mauro Maggioni
Office: 411 Krieger Hall
Web page: www.math.jhu.edu/˜ mauro
E-mail: myfirstname.maggioni@jhu.edu

Review problems

- Define the Dirichlet kernel and describe its role in studying Fourier series.
- Define approximations to the identity, and give an example (commenting on why it is indeed an approximation to the identity, and proving what you can). Is the Dirichlet kernel an approximation to the identity? Suppose you have an approximation to the identity $\{K_n\}$ such that K_n is a trigonometric polynomial of degree at most n: sketch how you could use this to prove completeness of the Fourier basis on $[0,1]$ in $L^2([0,1])$.
- True or false: let f be continuous and periodic on $[0,1]$. Then $f(x)$ is the limit of its partial Fourier series $\sum_{k=-K}^{K}\hat{f}(k)e^{2\pi ikx}$, for every $x \in [0,1]$. If yes, sketch a proof, if not, provide a counterexample, and discuss further conditions on f that would ensure such pointwise convergence.
- What can you say about the regularity of the function $f(x) = \sum_{k\in\mathbb{Z}} \frac{2k-1}{|k|^3+1}e^{2\pi ix}$ on $[0,1]$? Is this function in L^2? is it bounded? how many derivatives does it have?
- Describe how the Fourier transform on $\mathbb{R}$ or on the circle may be used to solve the heat equation, and how it can be used to give regularity results on its solution.
- Consider the Fourier basis on $[0,1]$. Compute the Fourier coefficients of the function $f(x) = x$ on $[0,1]$. How fast are coefficients decaying? Does this decay match your expectations? Explain the decay of the Fourier coefficients for this function. Same questions for the Haar wavelet coefficients (the Haar wavelets on $[0,1]$ are simply the Haar wavelets on $\mathbb{R}$ that have support (completely) contained in $[0,1]$).
- Let f be a function with 2 derivatives, with f, f', f'' all in $L^2(\mathbb{R})$. Consider the linear and nonlinear M-term approximation errors, $\epsilon_{\mathcal{B}}^{lin}[M]$ and $\epsilon_{\mathcal{B}}^{nonlin}[M]$ respectively. Do you expect them to decay faster, as a function of M, when $\mathcal{B}$ is the (i) Fourier basis, (ii) a wavelet basis with sufficient moments (how many?). Answer the same questions when f has the properties above only on each of K intervals partitioning $\mathbb{R}$ (i.e. the intervals are disjoint and their union is $\mathbb{R}$), but f and/or any of its two derivatives may be discontinuous at the boundary of such intervals.
- Compute the Fourier transform $\hat{f}$ on $[0,1]^2$ of the indicator function of $[0,1]\times[0,1/2]$. What is the direction of fastest decay of $\hat{f}$? of slowest decay?
- On $[0,1]^2$ consider the indicator function f of the ball centered at $(1/2,1/2)$ and of radius $1/4$. Estimate its 2-dimensional Haar wavelet coefficients as accurately as you can. What is the decay of these coefficients? How do the linear and nonlinear approximation errors decay? If the image was corrupted by Gaussian noise of variance σ^2, how would you denoise, and how close do you expect the denoised image to be to the true image?