Homework 5 - due Wed. Mar. 5th
High-Dimensional Approximation, Probability, and Statistical Learning

Instructor: Mauro Maggioni
Office: 302D Whitehead Hall
Web page: https://mauromaggioni.duckdns.org
E-mail: mauro.maggioni *at* youknowwhat.edu

Homework Policies

As in the first homework set.

Assignment

Study/review the parts discussed in class of Chapter 4 and 5 of R. Vershynin's High-Dimensional Probability lectures notes/book.

Exercises

Exercise 1 (100pt). Let C be a convex polytope, i.e. a compact (i.e. closed and bounded) subset of $\mathbb{R}^D$ which is the intersection of a finite number of half-spaces. Let V be the set of vertices of C. Show that if X is subGaussian with $||X||_{\psi_2} \leq K$, then

$$\mathbb{P}(\sup_{z\in C} |\langle z, X\rangle| > t) \lesssim |V| e^{-\frac{t^2}{2MK^2}} ,$$

where $|V|$ is the number of vertices of C, and M is a constant (that may depend on C.

[Hint: first do this for $\sup_{z\in C}\langle z, X\rangle$. Observe that the sup is a max (why?) and that this maximum is necessarily attained at a vertex of C [to show this, use convexity to write any point $z \in C$ as $z = \sum_{v\in V} \lambda_v v$ so $\lambda_v \geq 0$ and $\sum \lambda_v = 1$, to show that $\sup_{z\in C}\langle z, X\rangle \leq \sup_{z\in V}\langle z, X\rangle$ (the inequality in the other direction is trivial)].

Deduce a bound for $\mathbb{E}[\sup_{z\in C} |\langle z, X\rangle|]$.

Note, in particular, that these bound do not depend on D.

Apply the above to the unit ball in the ℓ^1 norm in $\mathbb{R}^D$, i.e. $\{x \in \mathbb{R}^D : ||x||_1 \leq 1\} = \{x \in \mathbb{R}^D : \sum_{i=1}^D |x_i| \leq 1\}$.

Compare this bound with a bound (say, in expectation) for the case when C is replaced by the unit ball in the Euclidean norm ℓ^2 (note that C is not a polytope in this case, and so the above does not apply; however we have already computed properties of bounds on the expectations and the tails in this case....when?).