Homework 1 - due Wed. Feb. 5th
Mathematical and Computational Foundations of Data Science

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Homework Policies

Homework is due at the beginning of class, stapled, written legibly, on one side of each page only. Otherwise, it will be returned ungraded.
All answers should be fully motivated - the logical arguments and motivation matter at least as much as answering correctly. The lowest homework score will be dropped. No late homework will be accepted. JHU policies apply with no exceptions to cases of incapacitating short-term illness, or for officially recognized religious holiday. You may, and are encouraged to, discuss issues raised by the class or the homework problems with your fellow students and both offer and receive advice. However all submitted homework must be written up individually without consulting anyone else's written solution.

The submission of homework that require numerical work on a computer should include the following: printout of the code used to solve the problems, of its inputs and of its outputs. The code should be written clearly, and should be commented at least in such a way that the input/outputs of the code is clear. The specific outputs requested by the exercise should be discussed in your writeup, critically and as needed in order to answer specific questions in the problems.

Assignment

Review your linear algebra! Review carefully any concepts you are not completely familiar with, by going back to your linear algebra textbook if needed. The references I suggested in class were the book by G. Strang, the book by Trefethen and Bau, Halmos' book, and of course your linear algebra textbook. The connection between matrices, linear operators, bases, changes of bases are sometimes not emphasized enough in linear algebra courses, but are fundamental for this course. Go through examples and short exercises proposed in class and do them.
Topics: vector spaces, vectors and matrices, operations between them, linear operators, key subspaces associated with linear operators (kernel, range; rank); transposition and inversion.
Look ahead: norms, inner products; orthogonal matrices; projections; positive definite matrices; eigenvalues and eigenvectors. (These are non-exhaustive lists.)

Exercises

Exercise 1 (1.1 from Strang's book; 10pts). Give an example where a combination of 3 nonzero vectors in $\mathbb{R}^4$ is the 0 vector. Then write your example in the form $Ax = 0$. What are the shapes and sizes of A and x?

Exercise 2 (30pts). The **rank** of a $m \times n$ matrix A is the number of independent rows of the matrix, or, equivalently, the number of independent columns of the matrix. The **range** of A is the span of the columns of A. Show that Ax is always in the span of A.

- Construct a 4×4 matrix A of rank 2.
- Find the range of this matrix, and a basis for that range.
- Pick a vector b in the range of A. Does the system $Ax = b$ have a solution x?
- Pick a vector b not in the range of A. Does the system $Ax = b$ have a solution x?

- Find all solutions of $Ax = 0$. Show that the sets of solutions forms a subspace, which we call the **kernel** of A (denoted $\ker(A)$).

Exercise 3 (60pts). [Vandermonde Matrix]. Fix $\{x_i\}_{i=1}^m$ points in an interval I (e.g. $I = [-1, 1]$). Consider the map $\mathbb{R}^{n+1} \to \mathbb{R}^m$ defined by $c = (c_0, \ldots, c_n) \mapsto (p_c(x_1), \ldots, p_c(x_m))$, where $p_c = \sum_{l=0}^n c_l x^l$, i.e. from vectors of coefficients of polynomials of degree $\leq n$ to vectors $(p_c(x_i))_{i=1}^m$ of values of such polynomials.

- Show that this map is linear
- Show that this map is represented, upon choosing the standard basis in $\mathbb{R}^{n+1}$ and $\mathbb{R}^m$, by the $m \times n$ Vendermonde matrix

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & x_m^2 & \ldots & x_m^n \end{bmatrix}.$$

  Therefore, if $c = [c_0; c_1; \ldots; c_n]$ is the column vector of coefficients of p, i.e. $p(x) = c_0 + c_1 x + c_2 x^2 \cdots + c_n x^n$, then Ac gives the the sampled polynomial values, i.e. $(Ac)_i = p(x_i)$, for $i = 1, \ldots, m$.

- When is this linear map invertible?