Homework 2 - due Wed. Feb. 12th
Mathematical and Computational Foundations of Data Science

Instructor: Mauro Maggioni
Office: 302D Whitehead Hall
Web page: https://mauromaggioni.duckdns.org
E-mail: mauro.maggioni *at* youknowwhat.edu

Homework Policies. As in the first homework set.

Assignment

Continue reviewing linear algebra. Review carefully any concepts you are not completely familiar with, by going back to your linear algebra textbook if needed. The references I suggested in class were the book by G. Strang, the book by Trefethen and Bau, Halmos' book, and of course your linear algebra textbook. The connection between matrices, linear operators, bases, changes of bases are sometimes not emphasized enough in linear algebra courses, but are fundamental for this course. Go through examples and short exercises proposed in class and do them.
Topics this week: linear operators, key subspaces associated with linear operators (kernel, range; rank); transposition and inversion; norms, inner products, (orthogonal) projections.
Look ahead: projections; positive definite matrices; eigenvalues and eigenvectors. (These are non-exhaustive lists.)

Exercises

Exercise 1 (30pts). Prove that if O is an orthogonal $n \times n$ matrix, representing a linear map $\mathbb{R}^n \to \mathbb{R}^n$, and $||\cdot||$ is the Euclidean norm, then

- $||Ov|| = ||v||$ and $||Ov - Ow|| = ||v - w||$ for all $v, w \in \mathbb{R}^n$
- $\langle Ov, Ow \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{R}^n$

Then show if A is a linear map $\mathbb{R}^n \to \mathbb{R}^n$ then the following are equivalent (i.e. each one implies the other):

- $||Av|| = ||v||$ for every $v \in \mathbb{R}^n$;
- $\langle Av, Aw \rangle = \langle v, w \rangle$ for every $v, w \in \mathbb{R}^n$.

[*Hint*: The inner product $\langle v, w, \rangle$ can be written in terms of norms, in particular of $||v + w||$ and $||v - w||$]

[*Hint*: ...if you still need help, look up „polarization identity for inner products"]

Exercise 2 (30pts). We work in $\mathbb{R}^n$, and fix a vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_+$, i.e. with $\alpha_1, \dots, \alpha_n > 0$.

- Show that
$$\langle v, w \rangle_\alpha = \sum_{i=1}^{n} \alpha_i v_i w_i$$
defines an inner product on $\mathbb{R}^n$.
- Find an $n \times n$ matrix A_α (that depends on α) such that you may write the inner product above as $v^T A_\alpha w$ (thinking of vectors as column vectors).
- Write an expression for the norm $||\cdot||_\alpha$ associated with this inner product, in terms of standard coordinates of a vector.

- Draw a sketch of the unit sphere $\{||v||_\alpha = 1\}$ corresponding to this norm, for some fixed α and for, say, $n = 2$.

Exercise 3 (40 pts). Construct an orthonormal basis for the space $\mathcal{P}^2([0,1])$ of polynomials of degree up to 2 on the interval $[0,1]$:

- first prove that if u, v are two linearly independent vectors, and $||u|| = 1$, then $v' := v - \langle v, u\rangle u$ is orthogonal to u, and the span of u, v' is the same as the span of u, v. You can then normalize v'; let $v'' = v'/||v'||$.

- now prove that if w is a third vector not in the span of u and v, then $w' := w - \langle w, u\rangle u - \langle w, v''\rangle v''$ is orthogonal to u and v'', and therefore the two-dimensional subspace spanned by u and v. Of course then w' may be normalized to $w'' := w'/||w'||$. Note that continuing this gives the Gram-Schmidt orthogonalization procedure: given linearly independent $v_1, \dots, v_m$, by computing iteratively

$$w_1 := v_1/||v_1||, \dots, w_{i+1} = (v_{i+1} - \sum_{i'=1}^{i} \langle v_{i+1}, w_{i'}\rangle w_{i'})/||v_{i+1} - \sum_{i'=1}^{i} \langle v_{i+1}, w_{i'}\rangle w_{i'}||$$

  up to $i + 1 = m$, one obtains an orthonormal basis $w_1, \dots, w_m$ for the span of $v_1, \dots, v_m$. Geometrically: note that, at every stage, $\sum_{i'=1}^{i} \langle v_{i+1}, w_{i'}\rangle w_{i'}$ is the orthogonal projection of v_{i+1} onto the span of $w_1, \dots, w_i$, and that subtracting that from v_{i+1} yields the portion of v_{i+1} orthogonal to that subspace, which up to normalization becomes w_{i+1}.

- use the above twice on the basis vectors $1, x, x^2$ to orthogonalize x to 1 and then x^2 to the span of 1 and x, and obtain the desired orthonormal basis.

Thinking of $\mathcal{P}^2([0,1])$ as a subspace of $\mathcal{P}^3([0,1])$, you have now an orthonormal basis for the 3-dimensional subspace $\mathcal{P}^2([0,1])$ inside the four-dimensional space $\mathcal{P}^3([0,1])$. Compute the orthogonal projection of x^3 onto this subspace $\mathcal{P}^2([0,1])$.

Thinking of $\mathcal{P}^2([0,1])$ as a subspace of $\mathcal{C}([0,1])$, the space of continuous functions on $[0,1]$, you have now an orthonormal basis for the 3-dimensional subspace $\mathcal{P}^2([0,1])$ inside the infinite-dimensional space $\mathcal{C}([0,1])$. Compute the orthogonal projection of $\sin(x)$ onto $\mathcal{P}^2([0,1])$.