Homework 3 - due Wed. Feb. 19th
Mathematical and Computational Foundations of Data Science

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Homework Policies. As in the first homework set.

Assignment

Continue reviewing linear algebra. Review carefully any concepts you are not completely familiar with, by going back to your linear algebra textbook if needed. The references I suggested in class were the book by G. Strang, the book by Trefethen and Bau, Halmos' book, and of course your linear algebra textbook. The connection between matrices, linear operators, bases, changes of bases are sometimes not emphasized enough in linear algebra courses, but are fundamental for this course. Go through examples and short exercises proposed in class and do them.
Topics this week: norms, inner products, (orthogonal) projections; SVD; eigenvalues/eigenvectors.

Exercises

Exercise 1 (30 pts). Review the SVD on any of the suggested references.

- Find the SVD of the matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.
- What is the rank of A?
- Solve the system $Ax = b$ for $b = \begin{bmatrix} -3 & -6 \end{bmatrix}$ and for $b = \begin{bmatrix} -3 & 1 \end{bmatrix}$ using the SVD of A. Have you solved the linear system or a least squares problem?
- What are the eigenvalues of $A^T A$?

Exercise 2 (70 pts). Divide the unit circle $\mathbb{S}^1$ in n equal intervals, using the n equi-spaced points $\mathbb{S}^1_n := \{x_i\}_{i=0}^{n-1}$ listed clockwise. You may also think of this as starting with the interval $[0,1]$, discretized at equally-spaced points $x_0 = 0, x_1 = \frac{1}{n-1}, \dots, x_i = \frac{i}{n-1}, \dots, x_{n-1} = \frac{n-2}{n-1}$, and then bending the interval into a circle, glueing the points 0 and 1. Also we will consider the points as having indices modulo n, meaning that $x_{n+i} = x_i$. So, for example, $x_n = x_0 = 0$, $x_{n+1} = x_1$, $x_{-1} = x_{n-1}$.

We now consider functions f from the discretized circle $\mathbb{S}^1_n$ to $\mathbb{R}$. We can also think of these functions as functions on the discretized unit interval that "wrap around" when reaching 0 (from the right) and 1 (from the left). A function $f : \mathbb{S}^1 \to \mathbb{R}$ is easily represented as a n-dimensional vector, namely the vector of function values $(f(x_1), \dots, f(x_n))$. These functions form an n-dimensional vector space, which is $\mathbb{R}^n$, endowed with the standard inner product and Euclidean norm.

Given two functions $f_1, f_2 : \mathbb{S}^1 \to \mathbb{R}$ write down a formula for their inner product, and for the norm of one of them, and compare to both the Euclidean inner product in $\mathbb{R}^n$ and to the L^2 inner products we defined in class (for example on continuous functions, or square-integrable functions, on $[0,1]$).

Consider the function: $f_0 = (\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{n-k})$, and the n functions $f_i(x_k) = f(x_{k-i})$, for $i = 0, \dots, n-1$, and note that f_i is f_0 translated to the right on the interval (with "wrap-around"), or clockwise on $\mathbb{S}^1$, by i points. Plot (on $\mathbb{S}^1$ or on the discretized interval $[0,1]$, with the latter probably being pictorially easier) the graphs of f_1 and f_4 for $n = 6$ and $k = 3$.

Compute the mean μ of the functions $f_0, \dots, f_{n-1}$: $\mu = \frac{1}{n} \sum_{i=0}^{n-1} f_i$, and plot the graph of this function (also for $n = 6$ and $k = 3$).

Consider the centered functions $\overline{f}_i = f_i - \mu$, represented as vectors as above. Stack the vectors $\{\overline{f}_i\}_{i=0}^{n-1}$ as columns of a matrix $\overline{F}_{k,n} = (\overline{f}_0|\ldots|\overline{f}_{n-1}) \in \mathbb{R}^{n\times n}$, where as above $n = 6$ and $k = 3$. Describe the matrix that you obtain, and any special pattern you observe in this matrix, and how it would change upon changing n and k.

Write code to construct the matrix $\overline{F}_{k,n}$ and compute its SVD.

- What can you say about U and V? Are they the same/similar/not similar, and why?
- Plot several of the columns of U as functions on the discretized interval, and similarly for the columns of V (if you think it is necessary). How do they look like? Can you give an exact or approximate analytic expression for these eigenvectors?
- How do the eigenvectors change with varying n and k?
- What about the singular values? Plot them for multiple values of n and k and comment on how they behave for a fixed n and k (as a function of the index of the singular value), and across multiple choices of n and k.